Colouring Empires in Random Trees

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by Andrew Robert Allen McGrae

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Declarations

This thesis is submitted to the University of Liverpool in support for my application for admission to the degree of Doctor of Philosophy. No part of it has been submitted in support of another degree or qualification at this or any other institution of learning. Parts of this thesis appeared in the following refereed papers in which by own work was that of a full pro-rata contributor:

A. R. McGrae, M. Zito. Colouring random empire trees, *Mathmatical Foundations of Computer Science (MFCS 2008)*, pp. 516–526.

A. R. McGrae, M. Zito. The block connectivity of random trees, *The Electronic Journal of Combinatorics*, vol. 16, R8, 2009.

C. Cooper, A. R. McGrae, M. Zito. Martingales on trees and the empire chromatic number of random trees, *Fundamentals of Computation Theory (FCT 2009)*, pp. 74–83, 2009.

Abstract

In this thesis we study the empire colouring problem as defined by Percy Heawood in 1890 for maps whose dual planar representation is a tree and have empires formed from exactly r vertices.

We define the reduced graph of one such tree as being the graph formed by replacing each empire in the tree by a single vertex such that an edge is present between two vertices u and v of the reduced graph if there was an edge in the original tree joining a vertex belonging to the empire corresponding to u to a vertex belonging to the empire corresponding to v. We then give several results on a number of structural properties of these graphs assuming that the underlying tree is a random labelled tree, and compare them to those of other types of random graphs with the same number of edges.

The main contribution of this thesis is a set of results on the worst-case and average-case properties of the empire colourings of trees having an upper bound s on the number of distinct colours used. We first show that if each empire contains exactly r countries, the empire colouring problem can be solved using no more than $2r$ colours on maps whose dual planar graph is a tree. Furthermore we give an inductive method for building instances that require these many colours. Then, motivated by the results of an empirical investigation of a number of colouring heuristics, we study the proportion of trees on a given number of vertices for which the empire colouring problem can be solved with $s \leq 2r$ colours. We prove that for every fixed positive integer r there exists a very precise lower bound on s beneath which almost all trees will admit no r-empire s-colouring. For larger values of s we are

then able to give constant positive lower bounds on the probability that s colours are sufficient to colour a random tree.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Graph theory is the branch of discrete mathematics concerned with the structure and properties of *graphs*, mathematical entities which can be used as models for binary relationships between objects within a set. A graph $G = (V, E)$ consists of a pair of sets such that the elements of E are unordered pairs of elements of V. In this thesis the elements of V will be called *vertices* (singular: *vertex*) and the elements of E *edges*, however elsewhere elements of V may be referred to as *nodes* or *points* and elements of E as *lines*. In this thesis, unless otherwise stated, we will follow *Graph Theory* by R. Diestel [26] for all graph theoretic definitions and notations.

The map colouring problem is one of the most famous graph theoretic problems. In 1852, Francis Guthrie put forward the conjecture, now known as the *Four Colour Theorem*, that four colours are sufficient to colour any map in such a way that no two regions sharing a border of non-zero length are given the same colour. (But regions may be given the same colour if they only meet at a single point, for example the states of Utah, Colorado, New Mexico and Arizona at the Four Corners in the USA). Of course in practice cartographers will often use more colours than are necessary, possibly for aesthetic reasons

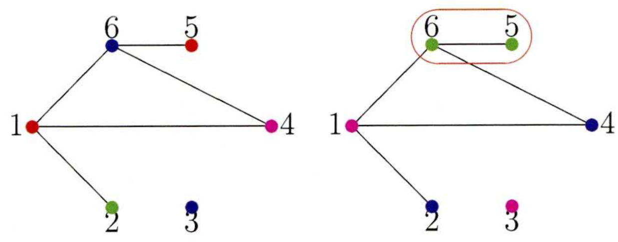

Figure 1.1: The first graph is properly coloured, however the second is not as adjacent vertices 5 and 6 are both green.

or because the colours represent something such as population levels.

This can be converted into graph theoretic terms by representing the regions as vertices and putting an edge between two vertices if and only if their corresponding countries share a border. It is quite easy to see that any map can be converted into a *planar* graph (i.e. a graph that can be drawn on a piece of paper without any line crossing) in this way and vice-versa. The resulting graph can be coloured by assigning a single colour to each vertex (in place of colours we can equivalently use the integers $1, 2, \ldots$), a *proper* colouring is one in which no two adjacent vertices are given the same colour. A colouring using exactly s colours is called an s-colouring, and $\chi(G)$, the *chromatic number* of G, is the smallest positive s for which G admits a proper s-colouring [17, 22].

Although the map colouring problem directly relates to planar graphs, there is no reason why graph colouring should be restricted to this type of structure. While non-planar graph colouring loses its connection to the original cartographical context it still has many practical applications. For example it can be used to produce a time-table for a number of end-of-year exams. Each exam could be seen as the vertex of a graph, with an edge joining two vertices if there are students taking both exams (and hence scheduling both at the same time would lead to a clash). If each possible timeslot is represented by a different colour then a proper colouring would correspond to a timetable with no clashes [9]. The book *Graph Colouring Problems* [49] by T. R. Jensen and B. Toft lists over 200 different problems related to graph colouring.

Obviously the chromatic number of an arbitrary graph may be quite large. For example, for any n, the complete graph K_n has chromatic number n. However, we can place restrictions on the graphs being considered and ask about the number of colours needed to colour properly any member of a given family of graphs. All planar graphs are four-colourable [8]; any graph with maximum degree Δ that is not a complete graph or odd-cycle can be coloured properly with Δ colours [19]; for large positive integers n, most of the graphs on n vertices containing less than $2.015n$ edges are colourable properly with just three colours [3].

The focus of this thesis is in fact on a particular colouring problem. Moving back to the planar graph colouring and its relationship to the colourability of cartographic maps, there exist certain countries in the world which are split into two or more non-contiguous regions. For example Alaska is considered to be part of the United States of America and Kaliningrad is part of Russia. To emphasise that these are in fact two parts of the same country it makes sense to give both regions the same colour. In the same paper in which he refuted a previous "proof" of the Four Colour Theorem [43] Percy John Heawood drew on this to suggest the following variation of graph colouring: suppose the vertices of a given graph are partitioned in a number of blocks, or *empires*, each consisting of r different vertices: we want to colour the graph using as few colours as possible in such a way that adjacent vertices receive different colours unless they belong to the same empire. Heawood was able to prove that for planar graphs and any $r \geq 2$, $6r$ colours are always sufficient, and in the case $r = 2$ there exist graphs requiring the full 12 colours. It has since been shown that for any $r \geq 2$ there are planar graphs requiring $6r$ colours [47].

The main aim of this thesis is to study such empire colourings on *trees* (i.e. connected graphs with no cycles). We will prove that if each empire consists of exactly $r \geq 1$ vertices then $2r$ colours are always sufficient to solve the given empire colouring problem, and there are trees requiring $2r$ colours. Furthermore we will investigate the colourability of n-vertex trees chosen uniformly at random. When considering randomly generated, rather than arbitrary graphs, the focus changes from seeking worst-case results to average-case results. Just a single example of a graph requiring s colours is enough to set s as a lower bound for the minimum number of colours needed to solve the empire colouring. However, when we consider random graphs, the question arises as to whether graphs requiring a relatively large number of colours are common or if there is something unusual about them (meaning that they will only very rarely turn up as the outcome of the random graph generation process). When we consider properties of random graphs, usually what we are concerned with is the probability of the graph possessing this property as the number of vertices tends to infinity. We will find exact and asymptotic expressions for the first two central moments of two random variables related to the number of s-*empire* colourings of an n-vertex random tree whose vertex set is partitioned into empires of size r. This in turn will enable us to prove that, for each $r \geq 1$, there exists a positive integer $s_r < r$ such that, for large n, almost all n-vertex trees need more than s_r colours, and then to give lower bounds on the proportion of such graphs that are colourable with $s > s_r$ colours.

We complete this introduction by presenting a short outline of the rest of the thesis. In Chapter 2 we will illustrate a number of "general-purpose" mathematical concepts and results that will be used throughout the thesis.

The initial part of Chapter 3 contains a precise definition of the problem we are studying and the graph models we will work with. The remainder of the chapter is, apparently, a detour from our main topic. We will study a number of combinatorial properties of a particular type of random graph. Properties of interest will be related to vertex degrees, connectivity, and the presence of certain small subgraphs. Finally the results obtained will be compared to similar results valid for other types of random graphs. Such investigation is not completely unrelated to the study of the empire colourability of random trees since the problem of colouring the empires in a random tree is equivalent to that of colouring the vertices of one of the random graphs studied here.

The last two chapters focus directly on the empire colouring problem on trees. In Chapter 4, we first describe our worst-case results and then analyse empirically a number of colouring heuristics. We prove that for all $r \geq 2$, there exists an algorithm that can properly colour all trees on empires containing exactly r countries each, using at most $2r$ colours and furthermore there exist trees where $2r$ colours are necessary. We also provide evidence supporting the claim that such worst-case results are rather pessimistic: there are heuristics (inspired by similar work on colouring other types of graphs [3, 65]) capable of colouring random trees with relatively few colours.

Finally in Chapter 5 we will present the main contribution of this thesis. Starting from a precise characterisation of the central moments of a random variable counting the number of s-empire colourings of a random tree with vertex set partitioned into empires of size r, we will be able to obtain upper and lower bounds on the probability that a random tree will have at least one proper colouring using s colours. For every r we find a value s_r such that almost all large random trees admits no empire colouring using at most s_r colours. Furthermore we complement this result by proving that if s is sufficiently larger than s_r (roughly about twice that value, for large values of r) then a non-negligible proportion of the set of tree on n vertices, with vertex set partitioned into empires of size r, can be coloured with s colours, provided n is large enough.

Chapter 2

Mathematical Preliminaries

In this Chapter we will cover a number of general mathematical results that will be used regularly throughout this thesis. We start with a quick review of well-known results and definitions in Linear Algebra, then we mention a number of relevant analytic and combinatorial concepts. In particular we describe a few simple approximations to the exponential and logarithm functions, harmonic numbers and multinomial coefficients which will be used in Chapter 5. The chapter is ended with a quick review of basic terminology and facts from Probability Theory.

2.1 Linear Algebra

In this thesis we will only use elementary linear algebra. All of our definitions can be found in standard textbooks such as [66]. If n and m are positive integers, an m by n *matrix* $\mathbf{A}$ is a table of (real) numbers with m *rows* and n *columns*, if $m = n$ then the matrix is *square*. The numbers in the matrix are called *elements*, $\mathbf{A}_{i,j}$ refers to the element in the i^{th} row and j^{th} column. If two matrices $\mathbf{A}$ and $\mathbf{B}$ are such that the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$, then the matrix $\mathbf{AB}$ is such that for any i and j, element $(\mathbf{AB})_{i,j}$ is equal to the sum $\sum_{k=1}^{n} \mathbf{A}_{i,k}\mathbf{B}_{k,j}$. The *transpose* of a matrix $\mathbf{A}$, denoted $\mathbf{A}^T$, is the matrix such that for any i and j, $\mathbf{A}^T_{i,j} = \mathbf{A}_{j,i}$, if $\mathbf{A} = \mathbf{A}^T$ the matrix is *symmetric*.

A matrix $\mathbf{A}$ in which all elements $\mathbf{A}_{i,j}$ with $i \neq j$ are zero is called a *diagonal* matrix. The *identity matrix* $\mathbf{I_m}$ is the m by m diagonal matrix in which all non-zero elements are equal to one. We define the matrices $\mathbf{One_m}$ and $\mathbf{Zero_m}$ as m by m matrices consisting entirely of ones or zeroes respectively.

A *row vector* is a 1 by n matrix and a *column vector* is an m by 1 matrix. If $\mathbf{v}$ is a column vector whose elements are not all zero and with $\mathbf{Av} = \lambda\mathbf{v}$ for some scalar value λ then $\mathbf{v}$ is called an *eigenvector* of $\mathbf{A}$ with *eigenvalue* λ. A set of vectors $\mathbf{v}^1, \ldots, \mathbf{v}^m$ are *linearly independent* if the only set of scalar values $\lambda_1, \ldots, \lambda_m$ such that $\sum_{i=1}^{m} \lambda_i \mathbf{v}^i = 0$ is the set where all $\lambda_i = 0$. Any n by n matrix $\mathbf{A}$ has up to n linearly independent eigenvectors, the eigenvalues of these define the *spectrum* of $\mathbf{A}$. In this thesis the spectrum of a matrix $\mathbf{A}$ will be denoted by Spec $\mathbf{A}$ and represented as a two row table containing the list of distinct eigenvalues of $\mathbf{A}$ in the top row, and their corresponding multiplicities (the number of independent eigenvectors corresponding to each given eigenvalue) in the bottom row.

The *determinant* is a function that associates a number, $|\mathbf{A}|$, to every n by n square matrix $\mathbf{A}$. If a matrix $\mathbf{A}$ is *diagonalisable*, meaning there exists an invertible matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{\Lambda}$ is diagonal, then $\mathbf{\Lambda}$'s diagonal elements (i.e. entries $\mathbf{\Lambda}_{i,i}$, for each $i \in \{1, \ldots . n\}$) are the eigenvalues of $\mathbf{A}$. Furthermore, in such a case, the determinant of $\mathbf{A}$ is equal to the product of its eigenvalues.

In Chapter 5 we will often manipulate various matrices which can be associated with graphs. In particular if G is a(n undirected) graph on n vertices, then its *adjacency matrix* $\mathbf{A}(G)$ is an n times n real symmetric matrix such that, for each $i, j \in \{1, \ldots, n\}$,

$$\mathbf{A}(G)_{i,j} = \begin{cases} 1 & \{i,j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

The *Laplacian matrix* $\mathbf{L}(G)$ of the graph G is an n times n real symmetric matrix such that, for each $i, j \in \{1, \ldots, n\}$,

$$\mathbf{L}(G)_{i,j} = \begin{cases} \deg_G(i) & i = j \\ -1 & i \neq j, \{i,j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

(here $\deg_G(i)$ is the number of vertices of G that are adjacent to vertex i).

2.2 Simple Analytical Preliminaries

In this thesis, the usual floor and ceiling notation $\lfloor x \rfloor$ and $\lceil x \rceil$ represent a real number x being rounded down or up respectively to the nearest integer. The *Kronecker delta* $\delta_{i,j}$ is a function such that $\delta_{i,j} = 1$ if $i = j$, zero otherwise. All logarithms are taken to base e.

We will use the standard Landau symbols (like $O(\phi)$ or $\Omega(\phi)$) in our asymptotic calculations. In particular, if f and g are two real functions, then $f(x) \sim g(x)$ represents the fact that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. In what follows we will often use the right arrow "$\rightarrow$" to denote limits. So, for instance, if f is a real-valued function defined on positive integers the expression "$f(n) \rightarrow c$ as n tends to infinity" (or "$n \rightarrow \infty$") is equivalent to $\lim_{n\to\infty} f(n) = c$.

2.2.1 Exponential and Logarithmic Functions

Here and in the rest of the thesis e^x or $\exp(x)$ denotes the exponential function (defined for any real number x), whereas $\log(x)$ is the natural logarithm of x, for $x > 0$. We now give a number of general results that will be used throughout this thesis.

Lemma 2.1 *For any real number z such that $|z| \leq \frac{4}{7}$,*

$$1 + z \leq \mathrm{e}^z \leq (1+z)(1+z^2).$$

Proof. We can express e^z as a Taylor series

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Hence, if $z > 0$ then $\mathrm{e}^z > 1 + z$. For the other inequality note that, since $z < 1$,

$$\begin{aligned} \mathrm{e}^z &\leq 1 + z + \frac{z^2}{2} + z^3\left(\frac{1}{3!} + \frac{1}{4!} + \ldots\right) \\ &= 1 + z + \frac{z^2}{2} + z^3\left(\mathrm{e} - \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{e}^z &\leq 1 + z + \frac{z^2}{2} + (\mathrm{e} - 2.5)z^3 \\ &\leq (1+z)\left(1 + \frac{z^2}{2}\right). \end{aligned}$$

If $z < 0$ we can write e^z as $\mathrm{e}^{-|z|}$. Set $x = |z|$, then $\mathrm{e}^{-x} > 1 - x$ is again obvious (implying $\mathrm{e}^z > 1 + z$). Also,

$$\mathrm{e}^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \sum_{k=3}^{\infty} \left(\frac{x^{2k-1}}{(2k-1)!} - \frac{x^{2k}}{(2k)!} \right).$$

We can thus write

$$\mathrm{e}^{-x} \leq 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!},$$

since the rightmost sum is positive for $x < 1$. Now, since $1 - \frac{x}{4} \geq \frac{3}{4}$ for $x < 1$, we can write

$$\frac{x^3}{3!} - \frac{x^4}{4!} = \frac{x^3}{3!} \left(1 - \frac{x}{4}\right) \geq \frac{x^3}{8}.$$

Therefore,

$$\mathrm{e}^{-x} \leq 1 - x + \frac{x^2}{2} - \frac{x^3}{8} = 1 - x + x^2 \left(\frac{1}{2} - \frac{x}{8}\right).$$

Since $x \leq \frac{4}{7}$, $\left(\frac{1}{2} - \frac{x}{8}\right) \leq (1 - x)$ and so

$$\mathrm{e}^{-x} \leq 1 - x + x^2(1 - x).$$

In other words, since $z = -x$, we can write $\mathrm{e}^{-x} = \mathrm{e}^z \leq (1 + z)(1 + z^2)$. ∎

Lemma 2.2 *For any real number y such that $|y| < \frac{1}{2}$,*

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{2} \leq \log(1 + y) \leq y - \frac{y^2}{2} + \frac{y^3}{3}.$$

In particular, if y tends to zero, we have $\log(1 + y) = y - \frac{y^2}{2} + o(y^2)$.

Proof. For any real number y with $|y| < 1$,

$$\log(1 + y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} y^k}{k}. \tag{2.1}$$

The upper bound on $\log(1+y)$ for $y > 0$ comes from discarding all terms for $k \geq 4$ (notice that since $y < 1$, the terms in the sum are strictly decreasing in size). We can also obtain a lower bound by discarding all terms of equation (2.1) for $k \geq 5$ giving

$$\log(1+y) \geq y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4},$$

which is greater than the stated lower bound.

Similarly, for negative y and $z = |y|$,

$$\log(1+y) = \log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}.$$

Hence the upper bound holds trivially for any non-negative $z < 1$. As for the lower bound, we can write

$$\begin{aligned}
\log(1-z) &\geq -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{1}{4}\sum_{k=4}^{\infty} z^k. \\
&= -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4(1-z)} \\
&\geq -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{2},
\end{aligned}$$

(where the second line uses the fact that $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, and the third uses that $z \leq 1/2$). The result follows by substituting back $y = -z$. ■

Let $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ be the n^{th} *Harmonic number*. The following result gives an estimate on the difference $H_n - H_m$ for sufficiently large integers n and m.

Lemma 2.3 *For all sufficiently large integers n and m, with $n > m$,*

$$H_n - H_m = \log\left(\frac{n}{m}\right) - \frac{n-m}{2mn} + O\left(\frac{1}{m^2}\right).$$

Proof. The result follows from the well-known asymptotic expression (see for instance [36, Equation (9.28)])

$$H_n = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + O\left(\frac{1}{n^6}\right),$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant. By the definition of $O()$ there must be positive a constant C such that

$$H_n - \log n - \gamma - \frac{1}{2n} \leq -\frac{1}{12n^2} + \frac{1}{120n^4} + \frac{C}{n^6} < -\frac{C_1}{n^2}$$

for some positive constant C_1 and n sufficiently large. Furthermore

$$H_n - \log n - \gamma - \frac{1}{2n} \geq -\frac{1}{12n^2} - \frac{C}{n^6} \geq -\frac{C_2}{n^2}$$

again for n sufficiently large. Therefore

$$\log n + \gamma + \frac{1}{2n} - \frac{C_2}{n^2} \leq H_n \leq \log n + \gamma + \frac{1}{2n} - \frac{C_1}{n^2}.$$

Now we look at $H_n - H_m$. First

$$H_n - H_m \leq \log\frac{n}{m} - \frac{n-m}{2nm} - \frac{C_1}{n^2} + \frac{C_2}{m^2} \leq \log\frac{n}{m} - \frac{n-m}{2nm} + \frac{C_2}{m^2}$$

for sufficiently large n and m, with $m \leq n$. Furthermore, by the same argument,

$$H_n - H_m \geq \log\frac{n}{m} - \frac{n-m}{2nm} - \frac{C_2}{n^2}.$$

The result follows. ∎

2.2.2 Multinomial Coefficients

For any positive integer n and non-negative integers $m_1, \ldots, m_t$ where $\sum_{i=1}^{t} m_i = n$, the *multinomial coefficient*

$$\binom{n}{m_1, \cdots, m_t} = \frac{n!}{\prod_{i=1}^{t} m_i!}$$

counts the number of ways in which n objects can be deposited into t bins such that for all $i \in \{1, \ldots, t\}$, bin i contains m_i objects. We always assume that $t \geq 2$. When $t = 2$, instead of writing $\binom{n}{m_1, m_2}$ we will use the equivalent *binomial coefficient* notation

$$\binom{n}{m_1} = \frac{n!}{m_1!\,(n - m_1)!}$$

(note that $m_2 = n - m_1$).

In Chapter 5 we will need to approximate particular multinomial coefficients near to "central" terms of the form

$$\binom{n}{\frac{n}{t}, \ldots, \frac{n}{t}}$$

(to avoid fiddling with integer parts assume that n/t is an integer). More specifically let c be a positive real number and consider multinomial coefficients of the form:

$$\binom{ctn}{m_1, \cdots, m_t}$$

(where we assume, for definiteness, that cn is an integer). To develop our approximations it is convenient to define, for each $i \in \{1, \ldots, t\}$, $x_i = m_i - cn$

and work with expressions like

$$\binom{ctn}{cn + x_1, \cdots, cn + x_t}$$

instead. Notice that $x_i \in \{-cn, \ldots, ctn - cn\}$ and $\sum_{i=1}^{t} x_i = 0$. For each tuple $(x_1, \ldots, x_t)$ satisfying these constraints define

$$f_n(x_1, \ldots, x_t) = \prod_{i=1}^{t} \frac{(cn)!}{(cn + x_i)!} \tag{2.2}$$

(the dependence of f on c will not be shown as in all relevant cases c will be a fixed constant). Notice that, since $x_t = -\sum_{i=1}^{t-1} x_i$, $f_n(x_1, \ldots, x_t)$ is really describing a function of $t-1$ (rather than t) variables (see Figure 2.1).

Furthermore we have

$$\binom{ctn}{cn, \ldots, cn} f_n(x_1, \ldots, x_t) = \binom{ctn}{cn + x_1, \ldots, cn + x_t}. \tag{2.3}$$

Thus $f_n(x_1, \ldots, x_t)$ can be used to express an arbitrary multinomial coefficient (on the right-hand side of equation (2.3)) in terms of the central one.

We will soon prove asymptotic bounds on $f_n(x_1, \ldots, x_t)$, but first we study a particular expression that will be used for this proof. Let h be a function defined on $\mathbb{Z}^+$, the set of positive integers. Consider the function $S^p_{n,h(n)}(x, y)$ defined as follows for $p \in \{1, 2\}$ and any integer x and y, with $y \in \{1, \ldots, x+1\}$:

$$\begin{aligned} S^p_{n,h(n)}(x, y) &= \frac{1}{(h(n)+x)^p} + \frac{1}{(h(n)+x-1)^p} + \ldots + \frac{1}{(h(n)+x-y+1)^p} \\ &= \sum_{k=0}^{y-1} \frac{1}{(h(n)+x-k)^p}. \end{aligned}$$

Lemma 2.4 *Let $n, x,$ and y be non-negative integers, with $y \in \{1, \ldots, x+1\}$.*

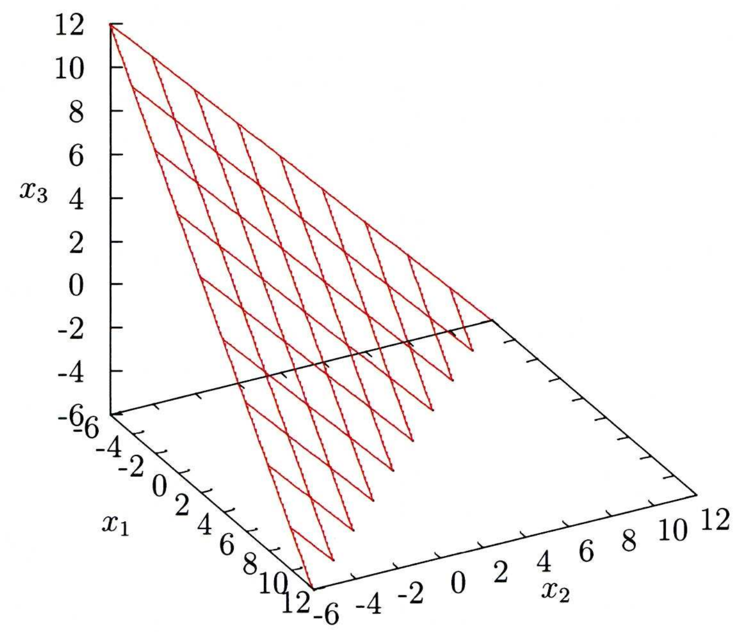

Figure 2.1: The red grid represents the set of triples (x_1, x_2, x_3) in the domain of f_6 for $c = 1$. In this case each x_i ranges in the set $R = \{-6, \ldots, 12\}$, $x_3 = -x_1 - x_2$, and values of x_1 and x_2 such that $x_1 + x_2 > 6$ are disallowed because they force x_3 out of R.

If $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $h(n) = \Theta(n)$, and $x = o(h(n))$, then there exists some positive constant C such that

$$S^2_{n,h(n)}(x, y) \leq \frac{Cy}{h(n)^2}$$

for n sufficiently large.

Proof. We can give an upper bound on $S^2_{n,h(n)}(x, y)$ by bounding above all terms in the sum by the term with the smallest denominator,

$$S^2_{n,h(n)}(x,y) \leq \frac{y}{(h(n)+x-y+1)^2} = \frac{y}{h(n)^2}\left(\frac{h(n)}{h(n)+x-y+1}\right)^2.$$

Since both x and y are $o(h(n))$, there exists a positive constant C such that

$$\frac{h(n)}{h(n)+x-y+1} < C$$

for all sufficiently large n, the result follows. ∎

Lemma 2.5 *Let n be a positive integer. Let c be a fixed positive real number, and assume that cn is an integer. Let t be a fixed integer, with $t \geq 2$, and let $\{x_1, \ldots, x_t\}$ be integers in the domain of f_n with* $\max|x_i| = o(n)$. *Then*

$$f_n(x_1,\ldots,x_t) = \exp\left\{-\sum_{i=1}^{t-1}\frac{x_i}{cn}\left(\sum_{l=1}^{i}x_l\right)\left(1+O\left(\frac{\max|x_i|}{cn}\right)\right)\right\}.$$

Roughly speaking the result implies, via (2.3), that if all x_i are much smaller than n then the multinomial coefficient

$$\binom{ctn}{cn+x_1,\cdots,cn+x_t}$$

is very close to the central term

$$\binom{ctn}{cn,\ldots,cn}.$$

Notice that the exponent of the exponential function in the statement above is always non-positive as

$$\sum_{i=1}^{t-1}x_i\left(\sum_{l=1}^{i}x_l\right) = \frac{\left(\sum_{i=1}^{t-1}x_i\right)^2}{2} + \frac{\sum_{i=1}^{t-1}x_i^2}{2}$$

and the square of any real number is always non-negative.

Proof of Lemma 2.5. We start by noticing that each tuple $(x_1, \ldots, x_t)$ in the domain of f_n must include both positive and negative values. Assume, without loss of generality, that $x_1, \ldots, x_{i_0}$ are all non-negative and $x_{i_0+1}, \ldots, x_{t-1}$ are all negative.

An equivalent way of stating the condition that $\sum_{i=1}^{t} x_i = 0$ is $x_t = -\sum_{i=1}^{t-1} x_i$. This can be rewritten as $x_t = x_t^- + x_t^+$ where x_t^- is the negative[1] part of x_t and x_t^+ is the positive part with the two terms defined as:

$$x_t^- = -\sum_{i=1}^{i_0} x_i, \quad x_t^+ = -\sum_{i=i_0+1}^{t-1} x_i.$$

The terms in the product given in equation (2.2) can be rewritten as

$$\begin{array}{ll} \displaystyle\prod_{k=0}^{x_i-1} \frac{1}{cn + x_i - k} & \text{For positive } x_i, \\ \displaystyle\prod_{k=0}^{|x_i|-1} cn - k & \text{For negative } x_i. \end{array}$$

The term for $i = t$ is equal to $\frac{(cn)!}{(cn+x_t)!}$ which can be rewritten as

$$\frac{(cn)!}{(cn + x_t^-)!}\frac{(cn + x_t^-)!}{(cn + x_t)!} = \left(\prod_{k=0}^{|x_t^-|-1} cn - k\right)\left(\prod_{k=0}^{x_t^+-1} \frac{1}{cn + x_t - k}\right).$$

Without loss of generality, we can therefore write $f_n(x_1, \ldots, x_t)$ as the product of two functions which we denote by $f^+(x_1, \ldots, x_t)$ and $f^-(x_1, \ldots, x_t)$ where

$$f^+(x_1, \ldots, x_t) = \left(\prod_{i=1}^{i_0}\prod_{k=0}^{x_i-1} \frac{1}{cn + x_i - k}\right)\left(\prod_{k=0}^{|x_t^-|-1} cn - k\right), \tag{2.4}$$

[1]Or, to be more precise, "non-positive".

(where the first product is over all those indices $i \in \{1, \dots, i_0\}$ for which x_i is strictly positive), and

$$f^-(x_1, \dots, x_t) = \left(\prod_{i=i_0+1}^{t-1} \prod_{k=0}^{|x_i|-1} cn - k \right) \left(\prod_{k=0}^{x_t^+-1} \frac{1}{cn + x_t - k} \right). \tag{2.5}$$

We will derive an asymptotic expression for $f_n(x_1, \dots, x_t)$ by studying f^+ and f^- in turn.

We first look at $f^+(x_1, \dots, x_t)$. By the way in which x_t^- is defined, there are exactly the same number of terms in either of the two products defining f^+. As such these can be merged together to give a single product as follows

$$\begin{aligned} f^+(x_1, \dots, x_t) &= \prod_i \prod_{k=0}^{x_i-1} \left(\frac{cn - \sum_{l=1}^{i-1} x_l - k}{cn + x_i - k} \right) \\ &= \prod_i \prod_{k=0}^{x_i-1} \left(1 - \frac{\sum_{l=1}^{i} x_l}{cn + x_i - k} \right). \end{aligned}$$

To simplify notations call $T_i = \sum_{l=1}^{i} x_l$. If $x_i \le \frac{4}{7t}(cn - t)$ for all $i \in \{1, \dots, i_0\}$ then $T_i \le \frac{4}{7}(cn - t)$, which in turn implies that for all $k \in \{0, \dots, x_i - 1\}$

$$\left| \frac{T_i}{cn + x_i - k} \right| \le \frac{4}{7}.$$

Thus, using repeatedly Lemma 2.1, and noticing that T_i does not depend on k, we have

$$f^+(x_1, \dots, x_t) \le \exp\left\{ -\sum_{i=1}^{i_0} T_i S^1_{n,cn}(x_i, x_i) \right\} \tag{2.6}$$

and, for large n,

$$\begin{aligned} f^+(x_1, \dots, x_t) &\ge \exp\left\{ -\sum_{i=1}^{i_0} T_i S^1_{n,cn}(x_i, x_i) \right\} \prod_i \prod_{k=0}^{x_i-1} \left(1 + \left(\frac{T_i}{cn + x_i - k} \right)^2 \right)^{-1} \\ &\ge \exp\left\{ -\sum_{i=1}^{i_0} T_i S^1_{n,cn}(x_i, x_i) \right\} \exp\left\{ -\sum_{i=1}^{i_0} (T_i)^2 S^2_{n,cn}(x_i, x_i) \right\} \end{aligned} \tag{2.7}$$

Next, we approximate $S^1_{n,cn}(x_i, x_i)$. Notice that, for each $i \in \{1, \ldots, i_0\}$, if $x_i > 0$ then $S^1_{n,cn}(x_i, x_i)$ is equal to $H_{cn+x_i} - H_{cn}$. Therefore, using Lemma 2.3, we have

$$S^1_{n,cn}(x_i, x_i) = \log\left(1 + \frac{x_i}{cn}\right) + O\left(\frac{x_i}{(cn)^2}\right).$$

As long as $x_i < \frac{cn}{2}$, by Lemma 2.2 we get

$$\begin{aligned} S^1_{n,cn}(x_i, x_i) &\geq \frac{x_i}{cn} - \frac{1}{2}\left(\frac{x_i}{cn}\right)^2 + \frac{1}{3}\left(\frac{x_i}{cn}\right)^3 - \frac{1}{2}\left(\frac{x_i}{cn}\right)^4 + h^1_n(x_i) \\ S^1_{n,cn}(x_i, x_i) &\leq \frac{x_i}{cn} - \frac{1}{2}\left(\frac{x_i}{cn}\right)^2 + \frac{1}{3}\left(\frac{x_i}{cn}\right)^3 + h^2_n(x_i) \end{aligned}$$

(here we assume that $h^1_n(x) = O\left(\frac{x_i}{(cn)^2}\right)$ and $h^2_n(x) = O\left(\frac{x_i}{(cn)^2}\right)$). From this, as $x_i = o(n)$ for all $i \in \{1, \ldots, i_0\}$, we get[2]

$$S^1_{n,cn}(x_i, x_i) = \frac{x_i}{cn} + O\left(\left(\frac{x_i}{cn}\right)^2\right)$$

since the term $(x_i/cn)^2$ dominates all others. Thus

$$\exp\left\{-\sum_{i=1}^{i_0} T_i S^1_{n,cn}(x_i, x_i)\right\} = \exp\left\{-\sum_{i=1}^{i_0} T_i \frac{x_i}{cn} + \sum_{i=1}^{i_0} T_i O\left(\left(\frac{x_i}{cn}\right)^2\right)\right\}. \qquad (2.8)$$

We now concentrate on the right-most term in the lower bound on f^+ in (2.7). By Lemma 2.4, for large n, $S^2_{n,cn}(x_i, x_i)$ can be bounded above by

$$\frac{Cx_i}{(cn)^2},$$

[2]If $0 < \lim x_i/n < +\infty$ then all terms in the bounds must be added up and we get much weaker bounds

$$K_1 + h^1_n(x_i) \leq S^1_{n,cn}(x_i, x_i) \leq K_2 + h^2_n(x_i).$$

for some positive constant C. Therefore

$$\exp\left\{-\sum_{i=1}^{i_0}(T_i)^2 S^2_{n,cn}(x_i,x_i)\right\} \geq \exp\left\{-C\sum_{i=1}^{i_0}(T_i)^2\left(\frac{x_i}{(cn)^2}\right)\right\}.$$

Notice that, since $\sum_{i\in I} x_i \leq |I| \max|x_i|$ (here I is any subset of $\{1,\dots,i_0\}$), the right-hand side in the expression above is at least as large as

$$\exp\left\{-Ct\max|x_i|\sum_{i=1}^{i_0} T_i\frac{x_i}{cn^2}\right\}. \tag{2.9}$$

Equation (2.8) and the bound (2.9) imply that

$$f^+(x_1,\dots,x_t) = \exp\left\{-\sum_{i=1}^{i_0} T_i\frac{x_i}{cn} + \sum_{i=1}^{i_0} T_i O\left(\frac{x_i \max|x_i|}{cn^2}\right)\right\}. \tag{2.10}$$

Next we turn to $f^-(x_1,\dots,x_t)$. As with x_t^-, x_t^+ is defined in such a way that there will be the same number of terms in each product and so the two products can be merged into one

$$\begin{aligned} f^-(x_1,\dots,x_t) &= \left(\prod_{i=i_0+1}^{t-1}\prod_{k=0}^{|x_i|-1} cn-k\right)\left(\prod_{k=0}^{x_t^+-1}\frac{1}{cn+x_t-k}\right) \\ &= \prod_{i=i_0+1}^{t-1}\prod_{k=0}^{|x_i|-1}\frac{cn-k}{cn+x_t-\sum_{l=i+1}^{t-1}|x_l|-k} \\ &= \prod_{i=i_0+1}^{t-1}\prod_{k=0}^{|x_i|-1}\frac{cn-k}{cn-T_i-k} \\ &= \prod_{i=i_0+1}^{t-1}\prod_{k=0}^{|x_i|-1}\left(1+\frac{T_i}{cn-T_i-k}\right), \end{aligned}$$

where $T_i = \sum_{l=1}^{i} x_l$. By repeatedly using Lemma 2.1, and noticing that T_i does not depend on k, we have

$$f^-(x_1,\dots,x_t) \leq \exp\left\{\sum_{i=i_0+1}^{t-1} T_i S^1_{n,cn}(-T_i,|x_i|)\right\} \tag{2.11}$$

and, for large n,

$$
\begin{aligned}
f^{-}(x_1,\ldots,x_t) &\geq \\
&\geq \exp\left\{\sum_{i=i_0+1}^{t-1} T_i S^1_{n,cn}\left(-T_i,|x_i|\right)\right\} \prod_{i=i_0+1}^{t-1}\prod_{k=0}^{|x_i|-1}\left(1+\left(\frac{T_i}{cn-T_i-k}\right)^2\right)^{-1} \\
&\geq \exp\left\{\sum_{i=i_0+1}^{t-1} T_i S^1_{n,cn}\left(-T_i,|x_i|\right)\right\} \exp\left\{-\sum_{i=i_0+1}^{t-1}(T_i)^2 S^2_{n,cn}\left(-T_i,|x_i|\right)\right\}.
\end{aligned}
\tag{2.12}
$$

We will approximate the bounds on $f^{-}(x_1,\ldots,x_t)$ as we did with $f^{+}(x_1,\ldots,x_t)$. Notice that, for each $i\in\{i_0+1,\ldots,t-1\}$, $S^1_{n,cn}\left(-T_i,|x_i|\right) = H_{cn-T_i}-H_{cn-T_i-|x_i|}$. Therefore, using Lemma 2.3, we have

$$
S^1_{n,cn}(-T_i,|x_i|) = \log\left(1+\frac{|x_i|}{cn-T_i-|x_i|}\right)+O\left(\frac{x_i}{(cn)^2}\right).
$$

Then, using Lemma 2.2 to approximate the logarithm, we get

$$
\begin{aligned}
S^1_{n,cn}(-T_i,|x_i|) &= \frac{|x_i|}{cn-T_i-|x_i|}-\frac{x_i^2}{2(cn-T_i-|x_i|)^2}+o\left(\frac{x_i^2}{(cn)^2}\right) \\
&= \frac{|x_i|}{cn}+O\left(\left(\frac{x_i}{cn}\right)^2\right).
\end{aligned}
$$

Thus, remembering that for all $i\in\{i_0+1,\ldots,t-1\}$, $x_i<0$, and therefore $|x_i|=-x_i$

$$
\exp\left\{\sum_{i=i_0+1}^{t-1} T_i S^1_{n,cn}\left(-T_i,|x_i|\right)\right\} = \exp\left\{-\sum_{i=i_0+1}^{t-1} T_i\frac{x_i}{cn}+\sum_{i=i_0+1}^{t-1} T_i O\left(\left(\frac{x_i}{cn}\right)^2\right)\right\}.
\tag{2.13}
$$

Finally, as before, we bound the right-most term in the lower bound on

f^-. By Lemma 2.4, for large n, $S^2_{n,cn}(-T_i, |x_i|)$ is at most

$$\frac{C_2|x_i|}{(cn)^2},$$

for some positive constant C_2. Therefore

$$\exp\left\{-\sum_{i=i_0+1}^{t-1} (T_i)^2 S^2_{n,cn}(-T_i, |x_i|)\right\} \geq \exp\left\{-C_2 \sum_{i=i_0+1}^{t-1} (T_i)^2 \frac{|x_i|}{(cn)^2}\right\}.$$

Notice that, since $\sum_{i\in I} x_i \geq -|I| \max |x_i|$ (here I is any subset of $\{i_0 + 1, \ldots, t-1\}$), the last expression is no larger than

$$\exp\left\{-C_2 t \max |x_i| \sum_{i=i_0+1}^{t-1} T_i \frac{|x_i|}{cn^2}\right\}. \tag{2.14}$$

Remembering that for all $i \in \{i_0 + 1, \ldots, t-1\}$, $x_i < 0$, and therefore $|x_i| = -x_i$, equation (2.13) and the bound (2.14) imply that

$$f^-(x_1, \ldots, x_t) = \exp\left\{-\sum_{i=i_0+1}^{t-1} T_i \frac{x_i}{cn} + \sum_{i=i_0+1}^{t-1} T_i O\left(\frac{x_i \max |x_i|}{cn^2}\right)\right\}. \tag{2.15}$$

We now have asymptotic bounds for both $f^+(x_1, \ldots, x_t)$ and $f^-(x_1, \ldots, x_t)$. By multiplying these together we get that

$$\begin{aligned} f_n(x_1, \ldots, x_t) &= \exp\left\{\left(-\sum_{i=1}^{i_0} T_i \frac{x_i}{cn} - \sum_{i=i_0+1}^{t-1} T_i \frac{x_i}{cn}\right)\left(1 + O\left(\frac{\max |x_i|}{cn}\right)\right)\right\} \\ &= \exp\left\{-\sum_{i=1}^{t-1} T_i \frac{x_i}{cn}\left(1 + O\left(\frac{\max |x_i|}{cn}\right)\right)\right\}. \end{aligned}$$

■

2.2.3 The Change of Variables Theorem

In Chapter 5, in order to get precise estimates on the higher moments of a particular random variable related to the number of proper empire colourings of a random tree, we will need to solve expressions in the form

$$\int_D \mathrm{e}^{-\frac{1}{2}\mathbf{y}\mathbf{A}\mathbf{y}^T} d\mathbf{y},$$

where D is some subset of $\mathbb{R}^m$, for some fixed m, and $\mathbf{A}$ is an m times m matrix with real coefficients. We will do this by changing the variables over which the function is integrated so as to give a more easily evaluated integral. The forthcoming presentation follows [7, Chapter XI, Section 32].

When the variables of a function to be integrated are changed, this can cause the content of the integration region D to be distorted. For this reason it is necessary to consider the extent of this distortion in order to calculate correctly the original integral.

Let $F_1(\mathbf{x})$ be a function integrated on t variables $\mathbf{x} = \{x_1, \ldots, x_t\}$ and $F_2(\mathbf{y})$ an equivalent function on variables $\mathbf{y} = \{y_1, \ldots, y_t\}$ where each x_i $(1 \leq i \leq t)$ can be expressed in terms of the $\mathbf{y}$ variables using a transformation $x_i = f_i(\mathbf{y})$. To transform $F_1(\mathbf{x})$ into $F_2(\mathbf{y})$ and preserve the content of the integration region we need the *Jacobian Matrix* corresponding to this transformation. This matrix is of the form (here f stands for the vector of functions $(f_1(\mathbf{y}),\dots,f_t(\mathbf{y}))$:

$$\mathbf{J}_f(\mathbf{y}) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \cdots & \frac{\partial x_1}{\partial y_t} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \cdots & \frac{\partial x_2}{\partial y_t} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{\partial x_t}{\partial y_1} & \frac{\partial x_t}{\partial y_2} & \cdots & \cdots & \frac{\partial x_t}{\partial y_t} \end{pmatrix}.$$

Thus a definite integral involving $F_1(\mathbf{x})$ can be transformed into one involving $F_2(\mathbf{y})$ using the determinant of $\mathbf{J}$ to preserve the content of the integration region:

$$\int_D F_1(\mathbf{x})d\mathbf{x} = \int_{f^{-1}(D)} F_2(\mathbf{y})|\mathbf{J}_f(\mathbf{y})|d\mathbf{y}. \tag{2.16}$$

The following result is the form of the change of variable theorem that will be needed in Chapter 5.

Lemma 2.6 *For each positive integer m, if $\mathbf{A}$ is an m times m non-singular positive-definite real symmetric matrix with eigenvalues $\lambda_1,\dots,\lambda_m$, then*

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{y}\mathbf{A}\mathbf{y}^T} d\mathbf{y} = (2\pi)^{\frac{m}{2}} \prod_{i=1}^{m} \frac{1}{\sqrt{\lambda_i}}.$$

Proof. By the *eigen decomposition theorem* (see [12, pp 161–162]), if $\mathbf{A}$ is a square matrix, $\mathbf{B}$ is a matrix of eigenvectors of $\mathbf{A}$ and $\mathbf{\Lambda}$ is a diagonal matrix with the corresponding eigenvalues on the diagonal then

$$\mathbf{A} = \mathbf{B}\mathbf{\Lambda}\mathbf{B}^{-1}. \tag{2.17}$$

Furthermore, if $\mathbf{A}$ is symmetric then $\mathbf{B}$ is an orthogonal matrix (and therefore $\mathbf{B}^{-1} = \mathbf{B}^T$). Note that $\mathbf{z} = \mathbf{y}\mathbf{B}$ defines a particular variable transformation

$f_{\mathbf{B}}$. Therefore, according to the method described above, we may write

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{y}\mathbf{A}\mathbf{y}^T} d\mathbf{y} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{y}\mathbf{B}\boldsymbol{\Lambda}\mathbf{B}^T\mathbf{y}^T} d\mathbf{y} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{z}\boldsymbol{\Lambda}\mathbf{z}^T} |\mathbf{J}_{f_{\mathbf{B}}}(\mathbf{z})| d\mathbf{z}.$$

However, since $\mathbf{B}$ is orthogonal, the Jacobian determinant is 1 and so

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{y}\mathbf{A}\mathbf{y}^T} d\mathbf{y} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{z}\boldsymbol{\Lambda}\mathbf{z}^T} d\mathbf{z}.$$

This can now be converted to a symmetric equation using the diagonal matrix $\mathbf{L}$ in which each diagonal element is equal to 1 divided by the square root of an eigenvalue of $\mathbf{A}$. Clearly $\mathbf{L}\boldsymbol{\Lambda}\mathbf{L}^T = \mathbf{I}$ and so

$$\begin{aligned}
\int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{z}\boldsymbol{\Lambda}\mathbf{z}^T} d\mathbf{z} &= |\mathbf{J}| \int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{w}\mathbf{L}\boldsymbol{\Lambda}\mathbf{L}^T\mathbf{w}^T} d\mathbf{w} \\
&= |\mathbf{J}| \int_{\mathbb{R}^m} e^{-\frac{1}{2}\mathbf{w}\mathbf{I}\mathbf{w}^T} d\mathbf{w} \\
&= |\mathbf{J}| \int_{\mathbb{R}^m} e^{-\frac{1}{2}\sum \mathbf{w}_i^2} d\mathbf{w}. \qquad (2.18)
\end{aligned}$$

where $\mathbf{w} = \mathbf{z}\mathbf{L}^{-1}$ and $|\mathbf{J}|$ is the Jacobian determinant of this transformation. Note that $\mathbf{L}^{-1}$ is the diagonal matrix with $(\mathbf{L}^{-1})_{i.i} = \sqrt{\lambda_i}$, the Jacobian matrix of the transformation is therefore equal to $\mathbf{L}$ and thus

$$|\mathbf{J}| = \prod_{i=1}^{m} \frac{1}{\sqrt{\lambda_i}}. \qquad (2.19)$$

Finally, it is well-known that

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}\sum \mathbf{w}_i^2} d\mathbf{w} = (2\pi)^{\frac{m}{2}}. \qquad (2.20)$$

The result now follows from (2.18) and (2.20). ■

2.3 Probability Theory

For all common probability theoretic terms see, for example, [38]. Here we quickly recall a few important definitions and results.

Let Q_n be an event describing a property of a random combinatorial structure depending on some integer parameter n. To say that Q_n holds *a.a.s.*, or *asymptotically almost surely* means that $\Pr[Q_n] \to 1$ as $n \to \infty$.

We will use $\mathrm{B}(n, p)$ and $\mathrm{Po}(\lambda)$ to denote the binomial and Poisson probability distributions, respectively. If $\mathcal{L}$ is a probability distribution then $X \approx \mathcal{L}$ denotes the fact that X is a random variable with distribution $\mathcal{L}$. The reader is referred to [38, Chapter VII] for the definitions of various modes of convergence that are relevant to sequences of random variables. Here we just introduce a couple of notations that will be used in the rest of the thesis. More specifically, if $X_1, X_2, \ldots$ is a sequence of random variables, $X_n \xrightarrow{\mathrm{D}} Y$ (resp. $X_n \xrightarrow{\mathrm{D}} \mathcal{L}$) denotes the fact that the sequence X_n *converges in distribution* to the random variable Y (resp. to a(ny) random variable Y with distribution $\mathcal{L}$).

If X is a discrete random variable with values $x_1, x_2, \ldots, x_n$ and for each $i \in \{1, \ldots, n\}$, $\Pr[X = x_i] = p_i$, then the *first moment* or *expectation* of X, $\mathbf{E}X$, can be computed as the sum

$$\mathbf{E}X = \sum_{i=1}^{n} p_i x_i. \tag{2.21}$$

The k^{th} *moment* of X, denoted $\mathbf{E}X^k$ is the expectation of the k^{th} power of X. In symbols we write

$$\mathbf{E}X^k = \sum_{i=1}^{n} p_i x_i^k. \tag{2.22}$$

The *variance* of X, represented $\mathrm{Var}X$ is a measure of how far the possible values of X are spread out from its expected value. Var$X$ can be calculated as the second moment of X minus the square of its expectation. The *standard deviation* of X is the square root of its variance.

The basic probabilistic tools that will be used in the forthcoming chapters to prove many of our results are *Markov's* and *Chebyshev's inequalities.*

Markov's inequality states that if X is a non-negative random variable with finite expectation, then

$$\Pr[X \geq t] \leq \frac{\mathbf{E}X}{t} \qquad \text{for any } t > 0. \tag{2.23}$$

Chebyshev's inequality states that for any random variable X where Var$X$ exists, for any $t > 0$

$$\Pr[|X - \mathbf{E}X| \geq t] \leq \frac{\mathrm{Var}X}{t^2}. \tag{2.24}$$

A consequence of Chebyshev's inequality is that if $\mathbf{E}X > 0$, then

$$\Pr[X = 0] \leq \frac{\mathrm{Var}X}{(\mathbf{E}X)^2}. \tag{2.25}$$

Since the variance of X can be rewritten as $\mathbf{E}X^2 - (\mathbf{E}X)^2$, the previous result can be expressed as:

$$\Pr[X = 0] \leq \frac{\mathbf{E}X^2}{(\mathbf{E}X)^2} - 1. \tag{2.26}$$

And so by finding the first and second moments of X it is possible to give bounds on the probability that X will be equal to zero.

The task mentioned above may be further simplified. If we know that the random variable X may be decomposed into the sum of elementary 0-1 random variables such that we can write $X = \sum_{\alpha \in A} I_\alpha$ then, by linearity,

$$\mathbf{E}X = \sum_{\alpha \in A} \Pr[I_\alpha = 1] \tag{2.27}$$

and in fact, for any $k \geq 1$,

$$\mathbf{E}X^k = \sum \Pr[I_{\alpha_1} = 1 \wedge \ldots \wedge I_{\alpha_k} = 1], \tag{2.28}$$

where the sum ranges over all possible k-tuples of values in A. Thus, for instance, when we look at the number of empire colourings of random trees, it will be convenient to compute the second moment using (2.28) rather than the definition (2.22).

In Chapter 5 it will also be convenient to use a stronger version of Chebyshev's inequality coming from the *Cauchy-Schwarz inequality*. Under the same assumptions of (2.24)

$$\Pr[X \neq 0] \geq \frac{(\mathbf{E}X)^2}{\mathbf{E}X^2} \tag{2.29}$$

(this is mentioned for instance in [48, Equation (3.3)]).

2.3.1 The Method of Moments

In Chapter 3 we will show that the number of certain structures within a random tree tends to a Poisson distribution. In this section we state two Theorems that will be used to do this.

Let X is a random variable. If the moments of X are all finite and every random variable with the same moments as X has the same distribution as X, then the distribution of X is said to be determined by its moments. The following theorem states an important property of all distributions determined by their moments.

Theorem 2.7 *Let Z be a random variable with distribution determined by its moments. If $X_1, X_2, \ldots$ are random variables such that for all $k > 0$, $\mathbf{E}(X_n)^k \to \mathbf{E}Z^k$ as $n \to \infty$ then X_n converges in distribution to Z.*

For any random variable S_n, let $\mathbf{E}(S_n)_t$ be the t^{th} *factorial moment* of S_n, defined as

$$\mathbf{E}(S_n)_t = \mathbf{E}[S_n(S_n - 1)\ldots(S_n - t + 1)].$$

The following variant of Theorem 2.7 is best suited to prove that the distribution of a certain sequence of random variables depending on some integer parameter n, approaches a Poisson distribution as n tends to infinity.

Theorem 2.8 *Given a random variable S_n depending on n, if $\lambda \geq 0$ is such that as $n \to \infty$*

$$\mathbf{E}(S_n)_t \to \lambda^t,$$

for all $t \geq 1$ then $S_n \xrightarrow{\mathrm{D}} \mathrm{Po}(\lambda)$.

Proofs of Theorems 2.7 and 2.8 can be found in, for example, [23, Theorem 4.5] and [48, Corollary 6.8] respectively.

The use of Theorems 2.7 or 2.8 aimed at finding the asymptotic distribution of some random variable, goes under the name the method of moments. Any application of the method of moments is made possible by the ability to estimate the moments of a particular random variable. Often the following approach works. If S is a random variable, and we can write $S = \sum_{\alpha \in A} Z_\alpha$, where, for each $\alpha \in A$, Z_α is a random indicator, then $\mathbf{E}(S)_t$ satisfies

$$\mathbf{E}(S)_t = \sum_{\alpha_1,\ldots,\alpha_t}^{*} \Pr[Z_{\alpha_1} = \ldots = Z_{\alpha_t} = 1], \tag{2.30}$$

where $\sum^{*}_{\alpha_1,\ldots,\alpha_t}$ is a sum over all sequences of distinct indices $\alpha_1, \ldots, \alpha_t \in A$.

Chapter 3

Models and Their Structural Properties

The aim of this Chapter is twofold. We first give a precise definition of the problem we are studying and the graph models we will work with. Then we investigate a number of combinatorial properties of a particular type of random graph. This investigation is related to the study of the empire colourability of random trees since the problem of colouring the empires in a random tree is equivalent to that of colouring the vertices of the random graphs studied here. Thus understanding the structure of such graphs may help in designing good colouring strategies.

3.1 The Empire Colouring Problem

In his 1890 paper [43] in which he refuted a previous "proof" of the Four Colour Theorem, Percy John Heawood mentioned an extension of the map colouring problem which has since come to be known as the *empire colouring problem*.

"The Problem, however, may be extended in another direction without departing from ordinary surfaces. In actual maps a "county" often consists of two detached portions, which are nevertheless required to be coloured similarly. Even one such divided county may require a fresh colour."

In the same paper, Heawood went on to show that if each county consists of at most r portions then $6r$ colours are sufficient to give a proper colouring of the map. It was later shown that for all $r > 1$ there exist maps requiring this many colours.

In graph-theoretic terms, given a graph G, a partition P of the vertices of G and a positive integer s, the empire colouring problem asks if there exists an s-colouring of G assigning distinct colours to all adjacent vertices belonging to different "*empires*" - sets of vertices in P - but using the same colour for all elements of each block of P. From now on an s-colouring will be an s-empire colouring unless otherwise stated.

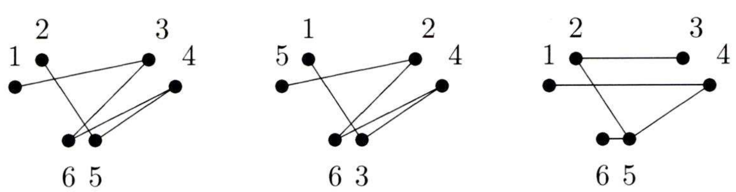

Figure 3.1: Simple examples of planar graphs whose vertex set is partitioned into three empires each containing two vertices (vertices that are closest to each other belong to the same empire).

Figure 3.1 shows three examples of graphs (in fact trees) whose vertex set has been partitioned into empires of size two. In the rest of the thesis we will be solely concerned with the case in which all empires contain the same number of vertices. Also, notice that the first two graphs have the same structure but different vertex labels. For the purpose of studying the empire colouring problem the actual labels of the vertices are unimportant in the following sense: for any colouring of the leftmost graph there is a corresponding colouring of the middle graph obtained by giving each vertex the same colour as the vertex in its place in the first graph.

Let $B = (B_i)_{i=1,\ldots,n/r}$ be the partition of $V(G)$ into blocks $B_1, B_2, \ldots$, such that B_k contains vertices $(k-1)r+1, (k-1)r+2, \ldots, kr$ of G. For each $i \in \{1, \ldots, r\}$, we will denote by k_i the vertex labelled $(k-1)r+i$ in B_k. Note that, for $r = 1$, the blocks contain just a single vertex, and, for $r > 1$ we always assume (even when we don't state it explicitly) that n/r is an integer[1]. From now on an instance of the empire colouring problem will be a triple (G, r, s) where G is a graph, and r and s are positive integers, with $r \leq n$ and such that n/r is a positive integer giving, respectively, the size of the empires in B, and the number of available colours. $\mathrm{C}(G, r, s)$ will denote the set of s-colourings for the given instance.

The empire colouring problem is a variant of the classical graph colouring problem that has received less attention and, nevertheless, has a number of interesting features. As it will become apparent very soon the problem reduces to classical graph colouring but the two problems are not equivalent. Also, the empire colouring problem is related to the colouring of graphs of given *thickness* (the reader is referred to the work of Hutchinson [46] for further details).

Given a graph G on n vertices, its *r-reduced graph*, $R_r(G)$ is a graph on vertices labelled $1, 2, \ldots, n/r$ having an edge connecting vertices i and j for each edge in G connecting a vertex $u \in B_i$ to a vertex $v \in B_j$. We will also say that G *reduces to graph* H or that H is the *reduced* graph of G, if $H = R_r(G)$, for some $r \geq 1$. Note that $R_r(G)$, in general, may contain

[1] A similar technical problem arises in the study of random r-regular graphs on n vertices (see for instance [48, Chapter IX]) where rn must be even for the graphs to be defined.

loops (if two vertices in B_i are adjacent) and parallel edges (if there are two different edges in $E(G)$ each joining a vertex in B_i to a vertex in B_j). Notice also that if $r = 1$, reducing each empire to a single vertex has no effect and so $R_1(G) \equiv G$. Figure 3.2 shows the 2-reduced graphs (vertex labels not displayed for clarity) corresponding to the graphs in Figure 3.1.

Figure 3.2: The 2-reduced graphs of the graphs in Figure 3.1: two of those graphs have the same reduced graph (the one on the left-hand side), the other graph reduces to a graph with a loop. Vertex labels have been omitted for clarity.

Given a colouring of $R_r(G)$, we can easily define a(n empire) colouring of G — for all $1 \leq k \leq \frac{n}{r}$ give every vertex in B_k the colour of vertex k in $R_r(G)$. Thus, finding an element of $\mathrm{C}(G, r, s)$ is equivalent to deciding whether $R_r(G)$ admits a (standard) colouring using s distinct colours. This fact was exploited by Heawood to prove that $6r$ colours are always enough to solve an instance of the empire colouring problem with empires consisting of at most r countries. More specifically, the r-reduced graph of a planar graph on n vertices has n/r vertices and, obviously, at most $3n-6$ edges. Thus the average degree of any induced subgraph of G is less than $6r$. This implies that there exists a vertex v in such graph of degree less than $6r$. Removing v leaves a smaller graph with the same small average degree property, which can be coloured recursively. Once this is done no more than $6r - 1$ colours will be in the neighbourhood of v. Hence v can be given a (spare) colour from a palette of $6r$ colours.

In the remainder of this chapter we analyse a number of additional structural properties of reduced graphs. Before doing that we need to make our probabilistic setting more precise.

3.2 Prüfer Codes

From now one we will concentrate mostly on instances of the empire colouring problem that consist of labelled trees. We next describe a well-known method of generating trees that will be useful in our analysis as it allows us to uniquely identify any tree with a string of integers.

Let $P = \{p_1, \dots, p_{n-2}\}$ be an ordered sequence of $n-2$ integers over $\{1, \dots, n\}$, called a *Prüfer code* [60], and let $\overline{K_n}$ be the edgeless graph on n vertices. We can associate a tree $T(P)$ to the given sequence by the following method:

Algorithm PruferToTree(P)

Set $T(P) = \overline{K_n}$.

For all i, set Used[i] = **false**.

for $i = 1$ to $n - 2$ **do**

Set v = first vertex that does not appear in $\{p_i, \dots, p_{n-2}\}$ and with Used[v] = **false**.

Add edge (v, p_i) to $T(P)$.

Set Used[v] = **true**.

end for

Set u and v as the two remaining vertices with Used[u] = Used[v] = **false**.

Add edge (u, v) to $T(P)$.

Return $T(P)$.

Notice that the process defines indeed a bijection between the set of strings of length $n-2$ over the alphabet $\{1,\ldots,n\}$ and the set of labelled trees on n vertices. Also, following this method, the degree of a vertex v in $\mathcal{T}_n(P)$ is equal to the number of times it appears in P, plus one.

3.3 Random Graphs

The main results in this thesis concern the size of $\mathrm{C}(G,r,s)$ under the assumption that G is a tree sampled uniformly at random among all trees on n vertices[2]. From now on $\mathcal{T}_n$ will denote a random labelled tree on n vertices (as defined for instance in [58]).

The rest of this chapter is devoted to the study of a number of structural properties of $R_r(\mathcal{T}_n)$. But what is $R_r(\mathcal{T}_n)$? Strictly speaking since $\mathcal{T}_n$ is a random graph so is its r-reduced graph. More precisely $R_r(\mathcal{T}_n)$ will be the typical element of a particular probability space $(\Omega,\mathcal{F},\Pr)$. In this space, Ω is the set of all graphs on n/r vertices. The second element of the triple $(\Omega,\mathcal{F},\Pr)$ is just the collection of all subsets of Ω. Finally Pr has the following definition (here Θ_n is the set of all trees on n vertices, and $H\in\Omega$):

$$\Pr[H]=\frac{|\{T\in\Theta_n : R_r(T)=H\}|}{n^{n-2}}.$$

Note that different trees may have the same r-reduced graph, but each tree corresponds to a single element of Ω. Conversely some graphs do not correspond to any tree, in which case $\Pr[H]=0$.

A number of remarks are in order. First, as a minor point, note that the definition of $(\Omega,\mathcal{F},\Pr)$ depends on positive integers n and r. Such depen-

[2]In this thesis we follow Janson *et al.* [48], and consider a random graph as a graph generated by some predefined random procedure, formalised as a probability space, and a mapping from this space into a family of graphs.

dence will usually be kept implicit. Second, the definition above does *not* assign the uniform measure to each element of Ω. It should be stressed that we are not interested in sampling uniformly at random from Ω. We mentioned that reduced graphs may help studying the empire colouring problem. This is true also if we are interested in random instances. However to be able to translate a result that, say, states that almost all (in the probabilistic sense) 2-reduced graphs of trees are 3-colourable to a similar empire colouring statement on trees we need to keep track, in probabilistic terms, of the proportion of trees that correspond to a particular reduced graph. This is the main reason behind our definition.

In the next section we will consider a number of properties of the r-reduced graphs of $\mathcal{T}_n$. Although not directly related to colouring these properties shed some light on the likely structure of $R_r(\mathcal{T}_n)$ and may help, in the future, to devise efficient ways of solving the empire colouring problem, at least on random trees.

3.4 Structural Properties of $R_r(\mathcal{T}_n)$

In this section we study the degree sequence, the connectivity and the number of copies of certain subgraphs in $R_r(\mathcal{T}_n)$. In particular we will prove that vertex degrees have (asymptotically) Poisson distribution. We will show that the connectivity of $R_r(\mathcal{T}_n)$ is a.a.s. either $r-1$ or r. Finally we will prove that the distribution of short cycles is also approximately Poisson and we will obtain results on the presence of small cliques in $R_r(\mathcal{T}_n)$. All our asymptotic results are valid in the limit as n, the number of vertices of the random tree, tends to infinity.

3.4.1 Vertex Degrees

If G is a graph, and v is a vertex in G, let $\deg_G(v)$ be the degree of v in G, the number of vertices adjacent to v and let $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$. For any non-negative integer k, let $N_k(G)$ be the number of vertices of degree k in G. We will first look at the distribution of $\deg_{R_r(\mathcal{T}_n)}(v)$ for any fixed vertex v of $R_r(\mathcal{T}_n)$. Then we will prove results on $\Delta(R_r(\mathcal{T}_n))$.

If T_n is a tree on n vertices, then the average degree of $R_r(T_n)$ is equal to the number of edges multiplied by two and divided by the number of vertices giving

$$\frac{2(n-1)}{\frac{n}{r}} = 2r - \frac{2r}{n} < 2r.$$

Also, the degree of any vertex in $R_r(T_n)$ must be at least r as each of the r vertices in an empire has degree at least 1 in the underlying tree. The next two lemmas give precise information about the distribution of $\deg_{R_r(\mathcal{T}_n)}(v)$ for any fixed vertex v of $R_r(\mathcal{T}_n)$.

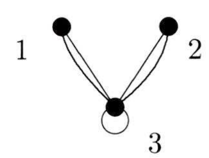

Figure 3.3: A 2-reduced graph of a tree in which a vertex has degree $n = 6$. The underlying tree is just a star centred at one of the vertices of empire 3.

Lemma 3.1 *Let positive integers k, r, and n be given, with $1 \le r \le k \le n-2+r$. For any $v \in V(R_r(\mathcal{T}_n))$ the number of trees T_n for which $\deg_{R_r(\mathcal{T}_n)}(v) = k$ is*

$$\binom{n-2}{k-r} r^{k-r} (n-r)^{n-2-k+r}.$$

Proof. Our proof generalises a well known result about trees (See, for instance [24]). We can count the number of trees reducing to graphs in which v has degree k by working on the associated Prüfer codes (see Subsection 3.2). As mentioned before, the degree of any vertex in T_n is equal to one plus the number of times that vertex appears in the Prüfer code.

If the vertex $v \in V(R_r(T_n))$ has degree k then the degrees of the vertices $v_1, \ldots, v_r \in V(T_n)$ representing the countries belonging to the empire denoted by v must sum to k. As each vertex v_i (for $i \in \{1, \ldots, r\}$) appears $\deg_{T_n}(v_i) - 1$ times in the Prüfer code P representing T_n, the vertices will appear a total of $k - r$ times in P.

There are $\binom{n-2}{k-r}$ possible choices for the elements of P which are equal to some v_i and r^{k-r} choices for which v_i each is equal to. The other $n-2-k+r$ elements may be equal to any of the $n-r$ other vertices in $V(T_n)$. The number of codes corresponding to trees for which $\deg_{R_r(T_n)}(v) = k$ is therefore:

$$\binom{n-2}{k-r} r^{k-r}(n-r)^{n-2-k+r}.$$

∎

We can now give the distribution of the vertex degrees within $R_r(\mathcal{T}_n)$. Recall from Section 2.3 that $X \approx \mathcal{L}$ denotes the fact that X is a random variable with distribution $\mathcal{L}$.

Lemma 3.2 *Let r be a fixed positive integer, and integer $n \geq r$ be such that n/r is a positive integer. Then*

$$\deg_{R_r(\mathcal{T}_n)}(v) - r \xrightarrow{\mathrm{D}} \mathrm{Po}(r)$$

as n tends to infinity, for any $v \in V(R_r(\mathcal{T}_n))$.

Proof. From Lemma 3.1 it follows that the probability that a vertex in the r-reduced graph of a random tree has degree k satisfies:

$$\Pr[\deg_{R_r(\mathcal{T}_n)}(v) = k] = \binom{n-2}{k-r}\left(\frac{r}{n}\right)^{k-r}\left(1-\frac{r}{n}\right)^{n-2-k+r}$$

for any $k \in \{r, \ldots, n-2+r\}$. In other words the random variable

$$\deg_{R_r(\mathcal{T}_n)}(v) - r \approx \mathrm{B}\left(n-2, \frac{r}{n}\right). \tag{3.1}$$

The result now follows, for fixed values of r, from the well known relationship between the binomial and the Poisson distribution (see, for instance, [38, Chapter III]). ∎

Vertices of minimum degree in $R_r(\mathcal{T}_n)$

As mentioned before, no vertex of $R_r(\mathcal{T}_n)$ can have less than r edges attached to it. The results in this section provide information about the number $N_r(R_r(\mathcal{T}_n))$ of vertices of minimum degree in $R_r(\mathcal{T}_n)$.

Lemma 3.3 *Let positive integers r, and n be given, with $r \leq n$ and such that n/r is a positive integer. Then*

$$\begin{aligned}
\mathbf{E}N_r(R_r(\mathcal{T}_n)) &= \frac{n}{r}\left(1-\frac{r}{n}\right)^{n-2} \\
\mathbf{E}N_r(R_r(\mathcal{T}_n))^2 &= \mathbf{E}N_r(R_r(\mathcal{T}_n)) + \left[\left(\frac{n}{r}\right)^2 - \frac{n}{r}\right]\left(1-\frac{2r}{n}\right)^{n-2}.
\end{aligned}$$

Proof. Let $\mathcal{E}_r(v)$ denote the event "$\deg_{R_r(\mathcal{T}_n)}(v) = r$", and let $m = \frac{n}{r}$. We can write

$$N_r(R_r(\mathcal{T}_n)) = \sum_{v=1}^{m} I_{\mathcal{E}_r(v)} \tag{3.2}$$

where $I_{\mathcal{E}_r(v)}$ is the random indicator for $\mathcal{E}_r(v)$. By Lemma 3.2, $\Pr[I_{\mathcal{E}_r(v)} = 1] = \left(1 - \frac{r}{n}\right)^{n-2}$. The result on $\mathbf{E}N_r(R_r(\mathcal{T}_n))$ follows.

Also, by (2.28), we may write

$$\mathbf{E}N_r(R_r(\mathcal{T}_n))^2 = \sum_{u,v=1}^{m} \Pr[I_{\mathcal{E}_r(v)} = 1, I_{\mathcal{E}_r(u)} = 1].$$

For any vertex $v \in V(R_r(\mathcal{T}_n))$, by Lemma 3.2 the number of trees reducing to graphs in which v has degree r is $(n-r)^{n-2}$. Given two distinct vertices $u, v \in V(R_r(\mathcal{T}_n))$ the number of trees reducing to graphs in which both u and v have degree r is equal to the number of Prüfer codes in which no vertex from B_u or B_v appears. That is, the number of distinct $(n-2)$-element strings with elements chosen from a set of $n-2r$ vertices. It is easy to see that there are $(n-2r)^{n-2}$ such strings. Hence

$$\mathbf{E}N_r(R_r(\mathcal{T}_n))^2 = \mathbf{E}N_r(R_r(\mathcal{T}_n)) + \left[\left(\frac{n}{r}\right)^2 - \frac{n}{r}\right]\left(1 - \frac{2r}{n}\right)^{n-2}.$$

■

Theorem 3.4 *Let r be a fixed positive integer and integer $n \geq r$ be such that n/r is a positive integer. Then*

$$N_r(R_r(\mathcal{T}_n)) = \frac{n}{r}\left(1 - \frac{r}{n}\right)^{n-2} + o(n) \quad a.a.s.$$

Proof. For positive integers r and n, with $n > r$,

$$\begin{aligned}
\left(1 - \frac{2r}{n}\right)^{n-2} &= \left(1 - \frac{r}{n}\right)^{n-2}\left(1 - \frac{r}{n-r}\right)^{n-2} \\
&= \left(1 - \frac{r}{n}\right)^{n-2}\left(1 - \frac{r}{n}\right)^{n-2}\left(1 - \left(\frac{r}{n-r}\right)^2\right)^{n-2} \\
&\leq \left(1 - \frac{r}{n}\right)^{2(n-2)}.
\end{aligned}$$

From this and Lemma 3.3, it is easy to see that

$$\mathrm{Var} N_r(R_r(\mathcal{T}_n)) \leq \mathbf{E} N_r(R_r(\mathcal{T}_n)).$$

Hence, by Chebyshev's inequality,

$$\Pr\left[|N_r(R_r(\mathcal{T}_n)) - \mathbf{E}N_r(R_r(\mathcal{T}_n))| \geq \sqrt{n\mathbf{E}N_r(R_r(\mathcal{T}_n))}\right] \leq \frac{1}{n}$$

which implies the stated result. ■

Maximum Degree in $R_r(\mathcal{T}_n)$

It has been shown by J. W. Moon in [57] that as n tends towards infinity, the expected maximum degree of a random tree on n vertices satisfies

$$\mathbf{E}\Delta(\mathcal{T}_n) \sim \frac{\log n}{\log\log n},$$

and that for almost all trees on n vertices, and any positive constant ϵ, $\Delta(\mathcal{T})$ satisfies:

$$(1-\epsilon)\frac{\log n}{\log\log n}\frac{\log\log\log n}{\log\log n} < \Delta(\mathcal{T}_n) - \frac{\log n}{\log\log n} < (1+\epsilon)\frac{\log n}{\log\log n}\frac{\log\log\log n}{\log\log n}. \tag{3.3}$$

These results can be used to give trivial upper and lower bounds on the maximum degree of the r-reduced graph of a random tree $\Delta(R_r(\mathcal{T}_n))$ — clearly the maximum degree must be at least as much as the maximum degree of the original tree, but no more than r times this. We can go further than this and show that the lower bound is often much closer to the truth and that for constant r, almost all trees have r-reduced graphs with maximum degree close to $\frac{\log n}{\log\log n}$.

Theorem 3.5 *Let r be a fixed positive integer and integer $n \geq r$ be such that n/r is a positive integer. Then for any constant $\epsilon > 0$,*

$$\frac{(1-\epsilon)\log n}{\log\log n} \leq \Delta(R_r(\mathcal{T}_n)) \leq \frac{(1+\epsilon)\log n}{\log\log n} \qquad a.a.s.$$

Proof. The result for $r = 1$ was proved by Moon in [57]. For $r \geq 2$, the lower bound follows from equation (3.3), as an empire may contain a vertex of maximum degree of the underlying random tree and $r-1$ vertices of degree one. For the upper bound first note that by equation (3.1),

$$\deg_{R_r(\mathcal{T}_n)}(v) - r \approx \mathrm{B}\left(n-2, \frac{r}{n}\right).$$

We also note that classical Chernoff bounds on the upper tail of a binomial random variable imply that (see [40]), if $S \approx \mathrm{B}(n,p)$, then

$$\Pr[S \geq k] \leq \left(\frac{np}{k}\right)^k \mathrm{e}^{k-np} \tag{3.4}$$

for any $k \geq pn$. Define k as

$$k = \frac{(1+\epsilon)\log n}{\log\log n} - r - 1.$$

Then, by (3.4)

$$\begin{aligned}\Pr\left[\deg_{R_r(\mathcal{T}_n)}(v) > \frac{(1+\epsilon)\log n}{\log\log n}\right] &\leq \left(\frac{r\log\log n}{(1+\epsilon)\log n}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}-r-1} \mathrm{e}^{\frac{(1+\epsilon)\log n}{\log\log n}-2r-1} \\ &\leq \left(\frac{r\mathrm{e}}{1+\epsilon}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}} \left(\frac{\log\log n}{\log n}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}-r-1} \end{aligned} \tag{3.5}$$

We can rewrite this by taking both a logarithm and an exponent of terms within the equation

$$\left(e^{\log\left(\frac{re}{1+\epsilon}\right)}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}}\left(\frac{\left(e^{\log\log\log n}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}}}{\left(e^{\log\log n}\right)^{\frac{(1+\epsilon)\log n}{\log\log n}}}\right)\left(\frac{\log n}{\log\log n}\right)^{r+1} =$$

$$= \left(e^{\log n}\right)^{\frac{(1+\epsilon)\log\left(\frac{re}{1+\epsilon}\right)}{\log\log n}}\left(\frac{\left(e^{\log n}\right)^{\frac{(1+\epsilon)\log\log\log n}{\log\log n}}}{\left(e^{\log n}\right)^{\frac{(1+\epsilon)\log\log n}{\log\log n}}}\right)\left(\frac{\log n}{\log\log n}\right)^{r+1}, \quad (3.6)$$

then by simplifying $e^{\log n}$ to n and grouping together all powers of n we can simplify (3.6) to

$$\frac{(\log n)^{r+1}}{n^{\frac{1+\epsilon}{\log\log n}\left(\log\log n-\log\log\log n-\log\left(\frac{re}{1+\epsilon}\right)\right)}(\log\log n)^{r+1}} \leq \frac{(\log n)^{r-1}}{n^{1+\epsilon-\frac{C\log\log\log n}{\log\log n}}} \quad (3.7)$$

for some positive constant C. Thus, by linearity of expectation, the expected number of vertices of degree greater than k in $R_r(\mathcal{T}_n)$ is at most

$$\frac{(\log n)^{r+1}}{n^{\epsilon-\frac{C\log\log\log n}{\log\log n}}}$$

and the stated upper bound follows by Markov's inequality. ■

3.4.2 Edges and Paths

In this subsection we will look at the presence of certain edges and paths in $R_r(\mathcal{T}_n)$. Such a graph obviously contains $\frac{n}{r}$ vertices, equal to the number of empires in the tree, and $n-1$ edges which may include some loops and parallel edges. The probability of a given edge e being in the reduced graph $R_r(\mathcal{T}_n)$ depends on the existence of a number of edges in the underlying tree. A similar assertion applies to longer acyclic paths.

We start by stating a result of J. W. Moon [56] that will be used in a number of proofs throughout this chapter.

Lemma 3.6 *Given positive integer n, let F be a forest on n vertices consisting of $t = n - |E(F)|$ trees $C_1, \dots, C_t$. The number of labelled trees on n vertices with F as a subgraph is exactly*

$$n^{n-|E(F)|-2} \prod_{i=1}^{t} |V(C_i)|.$$

The next result follows easily from Lemma 3.6 and the fact that there are n^{n-2} labelled trees on n vertices.

Corollary 3.7 *The probability that a random labelled tree on n vertices contains a given acyclic path on l edges is $\frac{l+1}{n^l}$.*

The r-reduced graph of a tree on n vertices contains an edge between two vertices u and v if there is at least one edge in the underlying tree connecting a vertex $u_i \in B_u$ to a vertex $v_j \in B_v$. Similarly, a path $\{v^{(1)}, \dots, v^{(l+1)}\}$ exists in $R_r(\mathcal{T}_n)$ if for any $1 \le i \le l$ there is an edge between $v^{(i)}$ and $v^{(i+1)}$, i.e. the subgraph of the underlying tree induced by $\{v_1^{(i)}, \dots, v_r^{(i)}, v_1^{(i+1)}, \dots, v_r^{(i+1)}\}$ contains at least one edge with one end-point in $\{v_1^{(i)}, \dots, v_r^{(i)}\}$ and the other one in $\{v_1^{(i+1)}, \dots, v_r^{(i+1)}\}$. In the next result let $\mathcal{E}(v^{(1)}, \dots, v^{(l+1)})$ denote the event that $R_r(\mathcal{T}_n)$ contains an acyclic path $v^{(1)}, \dots, v^{(l+1)}$.

Lemma 3.8 *Let r, and l be two fixed positive integers, and integer $n \ge r$ be such that n/r is a positive integer. Let $v^{(1)}, \dots, v^{(l+1)}$ be l fixed empires in $R_r(\mathcal{T}_n)$. Then, for sufficiently large n,*

$$\Pr\left[\mathcal{E}\left(v^{(1)}, \dots, v^{(l+1)}\right)\right] = \frac{r^{l+1}}{n^l} \left(\sum_{k=1}^{l} (r-1)^{k-1} \sum_{c_1, \dots, c_k} \prod_{j=1}^{k} c_j \right) (1 + o(1))$$

where the second summation is over all ways to choose k integers $c_i > 1$ summing to $l + k$.

Proof. Each of the l edges in the given path in $R_r(\mathcal{T}_n)$, may correspond to one among r^2 edges in the underlying random tree. This gives a total of r^{2l} possible subgraphs of the underlying random tree that will reduce to the path $v^{(1)}, \ldots, v^{(l+1)}$. We will call these subgraphs $p_1, \ldots, p_{r^{2l}}$. We have that

$$\Pr\left[\mathcal{E}\left(v^{(1)}, \ldots, v^{(l+1)}\right)\right] = \Pr\left[\bigcup_{i=1}^{r^{2l}} (p_i \in \mathcal{T}_n)\right]. \tag{3.8}$$

Boole's inequality tells us that (3.8) can be bounded above by

$$\sum_{i=1}^{r^{2l}} \Pr[p_i \in \mathcal{T}_n], \tag{3.9}$$

and Bonferroni inequalities (see, for example [29, p. 100]) give us a lower bound of

$$\Pr\left[\bigcup_{i=1}^{r^{2l}} (p_i \in \mathcal{T}_n)\right] \geq \sum_{i=1}^{r^{2l}} \Pr[p_i \in \mathcal{T}_n] - \sum_{i=1}^{r^{2l}-1} \sum_{j=i+1}^{r^{2l}} \Pr[(p_i \cup p_j) \in \mathcal{T}_n]. \tag{3.10}$$

Finding an exact value for (3.9) and an upper bound on

$$\sum_{i=1}^{r^{2l}-1} \sum_{j=i+1}^{r^{2l}} \Pr[(p_i \cup p_j) \in \mathcal{T}_n] \tag{3.11}$$

will therefore give us upper and lower bounds on the probability we are looking for.

We will first find the value of (3.9), by Lemma 3.6, the probability of a given subgraph p_i being in $\mathcal{T}_n$ depends on the sizes of the connected components of p_i. Let k be the number of components and for $1 \leq j \leq k$ let c_j be the number of vertices in component j. We can calculate the number of possible subgraphs satisfying given values of k and $c_1, \ldots, c_k$ by noticing that we need to pick one vertex out of r for each of the $l+1$ empires in the path, and also an additional one of the remaining $r-1$ vertices for each of the empires where two components meet. This gives a total of

$$r^{l+1}(r-1)^{k-1}$$

choices for any given value of k and $c_1,\ldots,c_k$. If $r>1$, we can now sum over all possible values of k and $c_1,\ldots,c_k$ to give that

$$\begin{aligned}\sum_{i=1}^{r^{2l}}\Pr[p_i\in\mathcal{T}_n] &= \sum_{k=1}^{l}\sum_{c_1,\ldots,c_k} r^{l+1}(r-1)^{k-1}\frac{1}{n^l}\prod_{j=1}^{k}c_j\\ &= \frac{r^{l+1}}{n^l}\sum_{k=1}^{l}(r-1)^{k-1}\sum_{c_1,\ldots,c_k}\prod_{j=1}^{k}c_j. \qquad (3.12)\end{aligned}$$

We now look for an upper bound on (3.11). By Lemma 3.6, the probability

$$\Pr[(p_i\cup p_j)\in\mathcal{T}_n]$$

is maximised when $(p_i\cup p_j)$ has only $l+1$ distinct edges and each of these edges is in its own component, hence we have the upper bound

$$\Pr[(p_i\cup p_j)\in\mathcal{T}_n]\leq\frac{2^{l+1}}{n^{l+1}}.$$

Noting that the sum in (3.11) has $\binom{r^{2l}}{2}$ terms, we now have an upper bound on (3.11) of

$$\sum_{i=1}^{r^{2l}-1}\sum_{j=i+1}^{r^{2l}}\Pr[(p_i\cup p_j)\in\mathcal{T}_n]\leq\binom{r^{2l}}{2}\frac{2^{l+1}}{n^{l+1}}. \qquad (3.13)$$

Following (3.10), we can now obtain a lower bound on (3.8) by subtracting (3.13) from (3.12)

$$\sum_{i=1}^{r^{2l}-1}\sum_{j=i+1}^{r^{2l}}\Pr[(p_i\cup p_j)\in\mathcal{T}_n]\geq\frac{r^{l+1}}{n^l}\left(\sum_{k=1}^{l}(r-1)^{k-1}\sum_{c_1,\ldots,c_k}\prod_{j=1}^{k}c_j\right)-\binom{r^{2l}}{2}\frac{2^{l+1}}{n^{l+1}}.$$

■

3.4.3 Vertex Connectivity

We next turn to the (vertex-)connectivity of the graph $R_r(\mathcal{T}_n)$. Following [26, p. 9 – 10], a non-empty graph G is connected if every pair of vertices in G is linked by a path, and for any positive integer d, G is d-connected if $|V(G)| > d$ and $G - X$ is connected for any set $X \subseteq V(G)$ with $|X| < d$. Obviously $R_r(\mathcal{T}_n)$ is connected since the original tree is connected and any path in that graph is preserved in $R_r(\mathcal{T}_n)$. In fact there are likely to be multiple paths between any two vertices $u, v \in R_r(\mathcal{T}_n)$ since each consists of r vertices in the underlying tree and each of the r^2 pairs of vertices $u_i \in B_u$, $v_i \in B_v$ are connected by a path in the tree. Note that by Theorem 3.4 $R_r(\mathcal{T}_n)$ a.a.s. has minimum degree r, hence the removal of the r neighbours of a minimum degree vertex disconnects the graph and so the connectivity of $R_r(\mathcal{T}_n)$ is a.a.s. at most r. In this section we will show that, for each fixed $r \geq 2$, the probability that $R_r(\mathcal{T}_n)$ is r-connected is bounded above by a positive quantity that approaches a value dependent on r (but independent of n) as n tends to infinity and, furthermore, that $R_r(\mathcal{T}_n)$ is a.a.s. $(r-1)$-connected. The first result will follow as a corollary from an application of the method of moments (see Section 2.3.1) to estimate the distribution of the number of vertices of degree r, in $R_r(\mathcal{T}_n)$, which are incident to at least one pair of parallel edges. The second one, for $r \geq 3$, will be proved by estimating the number of trees whose reduced graph would be disconnected by the removal of a set S of (at most) $r-2$ empires and showing that this number is small compared to n^{n-2}.

Connectivity Upper Bound

Let v be a vertex in $R_r(\mathcal{T}_n)$, we call v a *funny* vertex if $\deg_{R_r(\mathcal{T}_n)}(v) = r$ and v is incident to a double edge. Notice that the presence of a funny vertex v in $R_r(\mathcal{T}_n)$ implies that the graph is not r-connected, as the removal of the (at most) $r-1$ neighbours of v would leave v as an isolated vertex.

Lemma 3.9 *Let r and t be fixed positive integers with $r \geq 2$, and $n \geq r$ be such that n/r is a positive integer. For any set of t vertices $v^1, \ldots, v^t \in V(R_r(\mathcal{T}_n))$, the probability that $v^1, \ldots, v^t$ are funny vertices is*

$$\binom{r}{2}^t r^t \frac{(n-rt)^{n-2-t}}{n^{n-2}}(1+o(1))$$

as n tends to infinity.

Proof. For a vertex in $R_r(\mathcal{T}_n)$ to have minimum degree, each of its vertices must be a leaf in $\mathcal{T}_n$, the number of trees in which $v^1, \ldots, v^t$ are funny is therefore equal to the number of trees on $n - rt$ vertices

$$(n-rt)^{n-rt-2}$$

multiplied by the number of ways to add t groups of r vertices as leaves such that in each group two vertices have parents in the same empire. For each group there are $\binom{r}{2}$ choices for the two vertices that are to have parents in the same empire, and $r(n-rt)$ choices for the parent vertices. For each $1 \leq j \leq t$ the number of ways to choose the vertices within v^j and their parents is therefore

$$\binom{r}{2} r(n-rt).$$

We now must count the number of ways to choose parents for the remaining $r-2$ vertices in each group, we can give an upper bound by allowing any remaining vertex in v^j to choose any of the $(n-rt)$ vertices in the tree as its parent, giving a total of

$$\binom{r}{2} r(n-rt)^{r-1} \tag{3.14}$$

choices. This however, may overcount by counting trees more than once if there is more than one double edge incident to v^j. We therefore give a lower bound by counting only trees in which there is exactly one double edge and all other vertices have parents in different empires

$$\binom{r}{2} r(n-rt) \prod_{l=1}^{r-2} (n-rt-rl) = \binom{r}{2} r(n-rt)^{r-1}(1+o(1)). \tag{3.15}$$

It follows from (3.14) and (3.15) that the number of ways to add the rt vertices such that $v^1, \ldots, v^t$ are funny is

$$\binom{r}{2}^t r^t (n-rt)^{rt-t}(1+o(1)).$$

The result follows by multiplying this by the number of trees on $n-rt$ vertices and dividing by n^{n-2}. ■

Let $F(R_r(\mathcal{T}_n))$ be the number of funny vertices in $R_r(\mathcal{T}_n)$. Through an application of the method of moments, the next result describes the asymptotic distribution of $F(R_r(\mathcal{T}_n))$.

Theorem 3.10 *Let r be a fixed positive integer, with $r \geq 2$, and integer $n \geq r$ be such that n/r is a positive integer. Then*

$$F(R_r(\mathcal{T}_n)) \xrightarrow{\mathrm{D}} \mathrm{Po}\left(\binom{r}{2} \mathrm{e}^{-r}\right)$$

as n tends to infinity.

Proof. For fixed integer $t \geq 1$, let $\mathbf{E}(F(R_r(\mathcal{T}_n)))_t$ be the t^{th} factorial moment of $F(R_r(\mathcal{T}_n))$,

$$\mathbf{E}(F(R_r(\mathcal{T}_n)))_t = \sum_{v^1,\ldots,v^t}^{*} \Pr[v^1,\ldots,v^t \text{ are funny vertices}],$$

where the sum is over all t-tuples of distinct vertices $v^1,\ldots,v^t \in V(R_r(\mathcal{T}_n))$.

We can see that the number of ordered t-tuples is $\left(\frac{n}{r}\right)_t$, and by Lemma 3.9 the probability that all vertices are funny is

$$\binom{r}{2}^t r^t \frac{(n-rt)^{n-2-t}}{n^{n-2}}(1+o(1)).$$

The t^{th} factorial moment is therefore

$$\begin{aligned}
\mathbf{E}(F(R_r(\mathcal{T}_n)))_t &= \left(\frac{n}{r}\right)_t \binom{r}{2}^t r^t \frac{(n-rt)^{n-2-t}}{n^{n-2}}(1+o(1)) \\
&= \left(\frac{n}{r}\right)^t \binom{r}{2}^t r^t \frac{(n-rt)^{n-2-t}}{n^{n-2}}(1+o(1)) \\
&= \binom{r}{2}^t \frac{(n-rt)^{n-2-t}}{n^{n-2-t}}(1+o(1)) \qquad (3.16)
\end{aligned}$$

If $|z| \leq \frac{4}{7}$, Lemma 2.1 implies that

$$\mathrm{e}^z \leq (1+z)(1+z^2) \leq (1+z)\mathrm{e}^{z^2},$$

which with some rearranging gives us that

$$1+z \geq e^{z-z^2}. \qquad (3.17)$$

Since r and t are fixed, for sufficiently large n we can therefore bound (3.16) above by

$$\left(\binom{r}{2}\mathrm{e}^{-r}\right)^t (1+o(1)),$$

and below by

$$\left(\binom{r}{2}\mathrm{e}^{-r-\frac{r^2 t}{n}}\right)^t (1+o(1)).$$

The result follows by Theorem 2.8. ■

Let $\phi_{r,n}(k)$ denote the probability that $R_r(\mathcal{T}_n)$ contains $k \geq 0$ funny vertices. If $R_r(\mathcal{T}_n)$ contains one or more funny vertices, then the removal of the $r-1$ neighbours of one of these vertices would disconnect the graph. The probability that $R_r(\mathcal{T}_n)$ is r-connected can therefore be bounded above by $\phi_{r,n}(0)$. The following result is a direct consequence of Theorem 3.10.

Corollary 3.11 *Let r be a fixed positive integer with $r \geq 2$, and integer $n \geq r$ be such that n/r is a positive integer. Then the probability that the graph $R_r(\mathcal{T}_n)$ is r-connected is at most $\phi_{r,n}(0)$ and furthermore*

$$\phi_{r,n}(0) \to \mathrm{e}^{-\binom{r}{2}\mathrm{e}^{-r}}$$

as n tends to infinity.

Connectivity Lower Bound

Let m, r and d be fixed positive integers and set $n = mr$. Let G be a connected graph[3] on m vertices. If, for some $d < m-1$, G is not $(d+1)$-connected, then there exists a partition of $V(G)$ into non-empty sets A, B, and S, such that[4] $|S| = d$ and all edges in the graph are either internal to

[3]As we mentioned before, the r-reduced graph of a tree is always connected.

[4]We will only consider sets S containing exactly d vertices since if there is a smaller cut-set then any set formed from this by adding more vertices to it while leaving A and

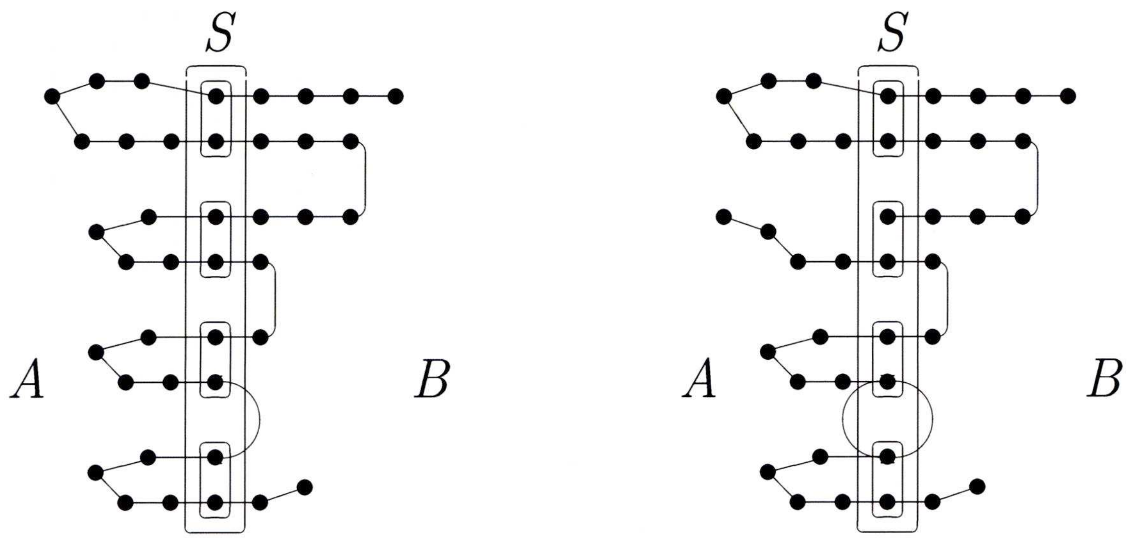

Figure 3.4: An example of (A, B, S)-arborescence (left) obtained from the r-reduced graph of a tree, with $r = 2$, $d = 4$, and $m = 20$. The sets of blocks S, A, and B are represented as sets of vertices. To avoid cluttering the picture only the four blocks of S have been represented as rectangles enclosing two vertices each. The vertices of all blocks in A are to the left of S, those blocks in B are to the right of S. The example on the right-hand side describes a more general case in which $F_A \cup F_B$ is not a tree.

one of the blocks or join a vertex in S to a vertex in either A or B. If G is the r-reduced graph of some graph H and G is not $(d+1)$-connected, the subgraph of H induced by the vertices in the blocks in $A \cup S$ (resp. $B \cup S$) will be denoted by F_A (resp. F_B) and will be such that each of its components contains at least one vertex in one of the blocks of S. Note that $F_A \cup F_B$ is not necessarily either connected or simple (see example on the right-hand side of Figure 3.4), however if $F_A \cup F_B$ is a tree (this is the case when $G = R_r(T_n)$), we call the pair (F_A, F_B) an (A, B, S)-*arborescence*. We obtain an upper bound on the number of trees on n vertices whose r-reduced graph would be disconnected by the removal of a set S of d vertices by estimating the total number of (A, B, S)-arborescences definable on a set of n vertices.

Given positive integers d, k, and n, positive integers $c_1, \ldots, c_k$ with

B non-empty will also disconnect the graph.

$\sum_{i=1}^{k} c_i = d$, let $h_{n,d}(c_1,\ldots,c_k)$ be the number of forests spanning a set V of $n+d$ vertices with k components such that, for each $i \in \{1,\ldots,k\}$, the i^{th} component contains $c_i > 0$ vertices in a given set $S \subseteq V$ of size d and x_i other vertices in $V \setminus S$. Then

$$h_{n,d}(c_1,\ldots,c_k) \leq \binom{d}{c_1,\ldots,c_k} \sum_{x_1,\ldots,x_k} \left(\binom{n}{x_1,\ldots,x_k} \prod_{i=1}^{k} (x_i+c_i)^{x_i+c_i-2} \right), \tag{3.18}$$

where the sum is over all k-tuples of non-negative integers $x_1,\ldots,x_k$ summing to n. Note that equation (3.18) overcounts slightly by counting some trees twice when $c_i = c_j$ for some $i \neq j$.

The total number of (A,B,S)-arborescences on a set of n vertices is at most

$$Z_{n,r,d} = \sum_{b=1}^{\frac{1}{2}\lfloor \frac{n}{r}-d \rfloor} \left(\binom{\frac{n}{r}}{\frac{n}{r}-b-d, b, d} \sum_{\mathbf{c}^A,\mathbf{c}^B} h_{n-br-dr,dr}(c_1^A,\ldots,c_{k_A}^A) h_{br,dr}(c_1^B,\ldots,c_{k_B}^B) \right), \tag{3.19}$$

where the inner sum is over all ways to choose two non-empty sequences of positive integers $c_1^A,\ldots,c_{k_A}^A$ and $c_1^B,\ldots,c_{k_B}^B$ adding up to dr. In the next section we will prove an upper bound on this quantity that is valid for fixed values of $r \geq 2$ and $d < r$, and sufficiently large values of n. This in turn leads to the following result, bounding the number of trees on $n = mr$ vertices whose r-reduced graph is $(r-1)$-connected.

Theorem 3.12 *Let r be a fixed positive integer with $r > 1$. There exists a positive constant*

$$C \leq (((r-2)r)!)^2 2^{2r(r-2)} (r-1)^{(r-1)r-2} r^{(r-1)^2-1}$$

such that, for any fixed $\epsilon \in \left(0, \frac{r-1}{r}\right)$, if n is sufficiently large, then the number *of trees T_n for which $R_r(T_n)$ is not $(r-1)$-connected is at most $Cn^{n-3+\epsilon}$.*

From Theorem 3.12, our result on the typical connectivity of $R_r(T_n)$ follows as a simple corollary.

Corollary 3.13 *For any fixed integer $r > 1$, the r-reduced graph of a random tree on n vertices is a.a.s. $(r-1)$-connected.*

Proof. By the previous Theorem, the number of trees on n vertices with r-reduced graphs that are not $(r-1)$-connected is at most $Cn^{n-3+\epsilon}$ for some constant C. The probability that a random tree will have an $(r-1)$-connected r-reduced graph is therefore at least

$$1 - \frac{C}{n^{1-\epsilon}}.$$

■

The rest of this section is devoted to the proof of Theorem 3.12. We start by working on $h_{n,d}(c_1, \ldots, c_k)$.

Lemma 3.14 *Let k and d be fixed positive integers. Then for any positive integer n, for all positive integers $c_1, \ldots, c_k$ with $\sum_{i=1}^{k} c_i = d$,*

$$\sum_{x_1,\ldots,x_k} \binom{n}{x_1, \ldots, x_k} \prod_{i=1}^{k} (x_i + c_i)^{x_i+c_i-2} \leq (n+d)^{n+d-2}.$$

Proof. Consider the sets of vertices $W = \{w_1, \ldots, w_n\}$, $S = \{u_1, \ldots, u_d\}$ and for $0 \leq i \leq d$ let $d_i = \sum_{j=1}^{i} c_j$. Then,

$$\binom{n}{x_1, \ldots, x_k} \prod_{i=1}^{k} (x_i + c_i)^{x_i+c_i-2}$$

counts the number of trees $T_1, \ldots, T_k$, where for $1 \leq i \leq k$, the tree T_i contains the vertices $u_{d_{i-1}+1}, \ldots, u_{d_i}$ and all vertices in W_i, given some arbitrary partition of W into k (possibly empty) subsets $W_1, \ldots, W_k$ where for each i, $|W_i| = x_i$. By summing over all $x_1, \ldots, x_k$ we consider all such partitions.

We can connect this sequence of trees by adding an edge $(u_{d_{i-1}+1}, u_{d_i+1})$ for every $1 \leq i \leq k-1$ to obtain a tree T with $n+d$ vertices. By construction, a different sequence of trees $T_1, \ldots, T_k$ leads to a different tree T . Thus we obtain that the number of different sequences of such trees $T_1, \ldots, T_k$ is less than or equal to the number of different trees T with $n+d$ vertices, which is

$$(n+d)^{n+d-2}.$$

■

Let r and d be fixed positive integers, with $r > 1$. For any positive integer n define

$$Y_{n,r,d}(a,b) = \binom{\frac{n}{r}}{a, b, d} (ar+dr)^{ar+dr-2}(br+dr)^{br+dr-2}.$$

By (3.18) and Lemma 3.14,

$$\left(\binom{\frac{n}{r}}{\frac{n}{r}-b-d, b, d} \sum_{\mathbf{c}^A, \mathbf{c}^B} h_{n-br-dr,dr}(c_1^A, \ldots, c_{k_A}^A) h_{br,dr}(c_1^B, \ldots, c_{k_B}^B)\right)$$

is at most

$$((dr)!)^2 C Y_{n,r,d}\left(\frac{n}{r} - b - d, b\right),$$

where the positive constant C is the number of ways to choose two non-empty sequences of positive integers $\mathbf{c}^A, \mathbf{c}^B$ each summing to dr. This is equal to

$$\left(\sum_{k=1}^{dr} \binom{dr-1}{k-1}\right)^2,$$

which is 2^{2dr-2} by the binomial theorem. In what follows we will consider

$Y_{n,r,d}(a,b)$ as defined on the set of positive integers a and b satisfying $a+b = \frac{n}{r} - d$.

Lemma 3.14 enables us to simplify our counting. The quantity $Z_{n,r,d}$ can be bounded above by $X_{n,r,d}\left(1, \frac{n}{2r} - \frac{d}{2}\right)$ where

$$X_{n,r,d}(b_1, b_2) = ((dr)!)^2 2^{2dr-2} \sum_{b=b_1}^{b_2} Y_{n,r,d}\left(\frac{n}{r} - b - d, b\right).$$

The remainder of our argument is a proof that this quantity is small compared with n^{n-2}.

To prove Theorem 3.12 we will split $X_{n,r,r-2}\left(1, \frac{1}{2}\left\lfloor\frac{n}{r} - r + 2\right\rfloor\right)$ into two parts:

$$X_{n,r,r-2}\left(1, \frac{1}{2}\left\lfloor\frac{n}{r} - r + 2\right\rfloor\right) \leq X_{n,r,r-2}(1, \lfloor n^\epsilon \rfloor) + X_{n,r,r-2}\left(\lfloor n^\epsilon \rfloor, \frac{1}{2}\left\lfloor\frac{n}{r} - r + 2\right\rfloor\right)$$

for some $\epsilon \in (0,1)$ to be chosen later. The following lemma shows that, for sufficiently large n, $Y_{n,r,d}(a,b)$ is maximised when either a or b is as large as possible. This fact will be used in turn to prove upper bounds on the two parts of $X_{n,r,r-2}\left(1, \frac{1}{2}\lfloor\frac{n}{r} - r + 2\rfloor\right)$.

Lemma 3.15 *Let d and r be fixed positive integers with $r \geq 3$, $d \leq r-2$, and integer $n \geq r$ be such that n/r is a positive integer. Then,*

$$Y_{n,r,d}(a+1, b-1) > Y_{n,r,d}(a,b)$$

for any integer a and b with $a > b \geq 1$, such that $a + b = \frac{n}{r} - d$.

Proof. For a fixed positive d,

$$\frac{Y_{n,r,d}(a+1,b-1)}{Y_{n,r,d}(a,b)} = \frac{\binom{\frac{n}{r}}{a+1,b-1,d}}{\binom{\frac{n}{r}}{a,b,d}} \frac{(ar+dr+r)^{ar+dr+r-2}}{(ar+dr)^{ar+dr-2}} \frac{(br+dr-r)^{br+dr-r-2}}{(br+dr)^{br+dr-2}}$$

$$= \frac{1}{a+1}\frac{(a+d+1)^{(a+1)r+dr-2}}{(a+d)^{ar+dr-2}}b\frac{(b+d-1)^{(b-1)r+dr-2}}{(b+d)^{br+dr-2}} \quad (3.20)$$

Define the function $f(x)$, for $x \geq 1$, as

$$f(x) = \frac{1}{x+1}\frac{(x+d+1)^{(x+1)r+dr-2}}{(x+d)^{xr+dr-2}},$$

then (3.20) is equal to

$$f(a)f(b-1)^{-1}.$$

The statement of this Lemma therefore holds if $f(x)$ is strictly monotone increasing for $x > 0$. The first derivative of $f(x)$ is equal to

$$\frac{(x+d+1)^{rx+rd+r-3}}{(x+d)^{rx+rd-1}}\times \frac{(x+1)\left(r(x+d)(x+d+1)\log\left(\frac{x+d+1}{x+d}\right)+2\right)-(x+d)(x+d+1)}{(x+1)^2}$$

which, for positive x and d, has the same sign as

$$r\log\left(1+\frac{1}{x+d}\right)+\frac{2(x+1)-(x+d)(x+d+1)}{(x+d+1)(x+d)(x+1)}.$$

Using Lemma 2.1 we can bound this below by

$$\frac{r}{x+d}-\frac{r}{(x+d)^2}+\frac{2(x+1)-(x+d)(x+d+1)}{(x+d+1)(x+d)(x+1)} =$$
$$= \frac{(r-1)x^3+((2d+1)r-(3d-1))x^2+((d^2+2d-1)r-(3d^2-2))x+((d+1)r-d(d+2))}{(x+d+1)(x+d)^2(x+1)}.$$

For positive x, d and $r \geq 3$ with $d \leq r-2$, every bracketed term in the last expression is non-negative and so $f'(x) > 0$ for all $x > 0$. Hence $f(x)$ is strictly monotone increasing for $x > 0$ and the result follows. ■

Proof of Theorem 3.12. For $r = 2$ the result is obvious since the reduced graph of any connected graph is itself connected. For $r > 2$, we give an upper bound on $Z_{n,r,d}$ and hence on the number of trees T_n for which the vertex set of $R_r(T_n)$ can be split in three sets A, B and S with $|S| = r - 2$, $|B| = b$ for some $b \in \left\{1, \ldots, \frac{1}{2}\lfloor \frac{n}{r} - r + 2\rfloor\right\}$, and $|A| = \frac{n}{r} - b - (r-2)$, and such that there are no edges connecting A to B. First note that

$$X_{n,r,r-2}\left(1, \frac{1}{2}\left\lfloor \frac{n}{r} - r + 2\right\rfloor\right) \leq X_{n,r,r-2}(1, \lfloor n^{\epsilon} \rfloor) + X_{n,r,r-2}\left(\lfloor n^{\epsilon} \rfloor, \frac{1}{2}\left\lfloor \frac{n}{r} - r + 2\right\rfloor\right).$$

Lemma 3.15 allows us to bound $X_{n,r,r-2}(1, \lfloor n^{\epsilon} \rfloor)$ above by making A as large as possible in each term

$$X_{n,r,r-2}(1, \lfloor n^{\epsilon} \rfloor) \leq (((r-2)r)!)^2 2^{2r(r-2)} n^{\epsilon} \left(\binom{\frac{n}{r}}{\frac{n}{r} - r + 1, 1, r-2} (n-r)^{n-r-2} ((r-1)r)^{(r-1)r-2} \right).$$

The multinomial coefficient $\binom{\frac{n}{r}}{\frac{n}{r} - r + 1, 1, r-2}$ is at most $\left(\frac{n}{r}\right)^{r-1}$, thus

$$\begin{aligned} X_{n,r,r-2}(1, \lfloor n^{\epsilon} \rfloor) &\leq (((r-2)r)!)^2 2^{2r(r-2)} n^{\epsilon} \left(\left(\frac{n}{r}\right)^{r-1} (n-r)^{n-r-2} ((r-1)r)^{(r-1)r-2} \right) \\ &\leq C n^{n-3+\epsilon} \end{aligned} \tag{3.21}$$

for some constant $0 < C \leq (((r-2)r)!)^2 2^{2r(r-2)} (r-1)^{(r-1)r-2} r^{(r-1)^2-1}$.

Next we look at

$$X_{n,r,r-2}\left(\lfloor n^{\epsilon} \rfloor, \frac{1}{2}\lfloor \frac{n}{r} - r + 2\rfloor\right),$$

this part of

$$X_{n,r,r-2}\left(1, \frac{1}{2}\lfloor \frac{n}{r} - r + 2\rfloor\right)$$

still contains a large number of terms, but each term is relatively small. By

Lemma 3.15 moving vertices from B to A will increase the size of

$$Y_{n,r}\left(\frac{n}{r}-b-(r-2),b,r-2\right).$$

We can therefore bound

$$X_{n,r,r-2}\left(\lfloor n^{\epsilon}\rfloor,\frac{1}{2}\lfloor\frac{n}{r}-r+2\rfloor\right)$$

above by

$$(((r-2)r)!)^2\,2^{2r(r-2)}\,\frac{n}{2r}\times$$

$$\left(\binom{\frac{n}{r}}{\frac{n}{r}-r-\lfloor n^{\epsilon}\rfloor+2,\lfloor n^{\epsilon}\rfloor,r-2}(n-r\lfloor n^{\epsilon}\rfloor)^{n-r\lfloor n^{\epsilon}\rfloor-2}(r\lfloor n^{\epsilon}\rfloor+r^2-r)^{r\lfloor n^{\epsilon}\rfloor+r^2-r-2}\right).$$

In the expression above, the multinomial coefficient is at most $\left(\frac{n}{r}\right)^{\lfloor n^{\epsilon}\rfloor+r-2}$, and thus we get (for n sufficiently large)

$$X_{n,r,r-2}\left(\lfloor n^{\epsilon}\rfloor,\frac{1}{2}\left\lfloor\frac{n}{r}-r+2\right\rfloor\right)\leq(((r-2)r)!)^2 2^{2r(r-2)}r^{r^2}r^{(r-1)n^{\epsilon}}n^{n-2+\epsilon(r^2-r-2)-(r-1-\epsilon r)n^{\epsilon}}.$$

For $r\geq 2$ and $0<\epsilon<\frac{r-1}{r}$, this means that

$$X_{n,r,r-2}\left(\lfloor n^{\epsilon}\rfloor,\frac{1}{2}\left\lfloor\frac{n}{r}-r+2\right\rfloor\right)\leq C'n^{n-3}. \tag{3.22}$$

for some constant $0<C'\leq(((r-2)r)!)^2 2^{2r(r-2)}r^{r^2}$. The result follows by adding together (3.21) and (3.22). ■

3.4.4 Cycles

A tree by definition contains no cycles. However, for $r\geq 2$, when an r-reduced graph is generated from a tree T_n it is possible that a cycle will be created from one or more paths within T_n. Two paths p_1 and p_2 in T_n describe a single path in $R_r(T_n)$ if there exists an empire B_v such that both p_1 and p_2 have end-points in $\{v_1, \ldots, v_r\}$. A path in T_n forms a cycle in $R_r(T_n)$ if both of its end-points are in the same empire. Given a collection of paths α in $\mathcal{T}_n$, we call α an *r-pre-cycle* (sometimes omitting r when such parameter is arbitrary, or clear from the context) of length k if the edges of α form a k-cycle in the r-reduced graph of $\mathcal{T}_n$. Figure 3.5 gives further illustration.

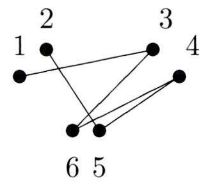

Figure 3.5: The 2-reduced graph of the tree in the picture contains three copies of the triangle spanning its three vertices. One copy is obtained as the union of the edge $\{2, 5\}$ and the path $(1, 3, 6)$ in T_n. The other two are each formed by three independent edges: they both contain $\{1, 3\}$, and $\{2, 5\}$, plus $\{4, 6\}$ in one case and $\{4, 5\}$ in the other one.

In what follows, let $X_k(G)$ denote the number of k-cycles (for $k \geq 1$) in a given graph G (where a 1-cycle is a loop and a 2-cycle a pair of parallel edges connecting two given vertices). Let $C_{m,k}$ be the set of all possible k-cycles on m labelled vertices. Note that $|C_{m,k}| = \binom{m}{k}$ (resp. $|C_{m,k}| = \frac{(m)_k}{2k}$), for $k \leq 2$ (resp. for $k > 2$). Formally, $C_{m,k}$ is a collection of graphs on m labelled vertices each containing exactly k edges arranged in a single undirected cycle. The $m - k$ isolated vertices are irrelevant to our treatment, hence, in what follows we will identify each element of $C_{m,k}$ with the particular cycle it contains.

We can find the expected number of k-cycles in $R_r(\mathcal{T}_n)$ by summing the probabilities of occurrence for each cycle in $C_{m,k}$. The probability that a given cycle $\gamma \in C_{\frac{n}{r},k}$ is formed by a collection of edges in $R_r(\mathcal{T}_n)$ depends on the way these edges are related in the tree. The *path-structure* of a collection of k edges in a tree is a sequence $I = (i_1, \ldots, i_k)$ of non-negative integers with $\sum_{l=1}^{k} l \cdot i_l = k$ where, for all $1 \leq l \leq k$, i_l is the number of paths of length l in the given collection of edges. Figure 3.6 represents graphically all valid path-structures for a set of three edges: (3,0,0) (three independent edges), (1,1,0) (one edge and a path of length two) and (0,0,1) (one path of length three).

Let $\mathcal{I}_k$ be the set of all k-tuples $(i_1, \ldots, i_k) \in \mathbb{N}^k$ such that $\sum_{l=1}^{k} l \cdot i_l = k$. Define $\lambda_{k,r}$ as

$$\lambda_{k,r} = \sum_{I \in \mathcal{I}_k} \binom{|I|}{i_1, \ldots, i_k} \frac{(r-1)^{|I|}}{2|I|} \prod_{l=1}^{k} (l+1)^{i_l}$$

(here $|I| = \sum_{l=1}^{k} i_l$).

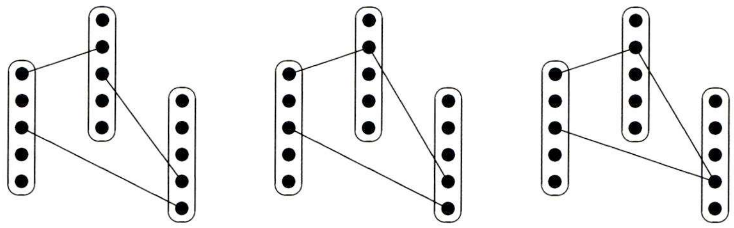

Figure 3.6: Possible ways in which three edges of a tree on $n = 15$ vertices may form a 5-pre-cycle reducing to a triangle in the 5-reduced graph of the structure (all other edges in the underlying tree are omitted for clarity).

Lemma 3.16 *Let k, r, and n be positive integers, with $r \geq 2$ and such that $\frac{n}{r}$ is a positive integer not smaller than k. Let T_n be a tree on n vertices. The number of possible r-pre-cycles consisting of k independent edges in T_n is*

$$\frac{\left(\frac{n}{r}\right)_k}{2k} (r(r-1))^k.$$

Proof. For $k \leq 2$ there are $|C_{n/r,k}| = \binom{n/r}{k}$ ways to choose the participating empires. We then choose two distinct vertices within each empire to be the start and end points of the k edges, there are $(r(r-1))^k$ ways to choose these vertices, but we must divide this by two as when $k = 1$ (resp. $k = 2$) a pre-cycle with edge $\{u_i, u_j\}$ (resp. edges $\{u_{i_1}, u_{j_1}\}$, $\{u_{i_2}, u_{j_2}\}$) is identical to a pre-cycle with edge $\{u_j, u_i\}$ (resp. edges $\{u_{i_2}, u_{j_2}\}$, $\{u_{i_1}, u_{j_1}\}$).

For $k > 2$, we first need to choose an ordered list of k vertices within the reduced graph on which the cycle is to be built, this can be done in $\left(\frac{n}{r}\right)_k$ different ways. Notice however that reversing the order of the vertices or choosing a different starting point does not change the cycle (e.g. $\{1, 2, 3, 4, 5\}$ is the same pre-cycle as $\{3, 4, 5, 1, 2\}$ or $\{4, 3, 2, 1, 5\}$) and so we must divide by $2k$ to avoid repetition. Having chosen a cycle on the empires, there are then $(r(r-1))^k$ choices for the vertices in the tree that are incident to the edges. ■

Theorem 3.17 *For fixed integers $r > 1$ and $k \geq 1$ and integer $n \geq r$ tending towards infinity,*

$$\mathbf{E}X_k(R_r(\mathcal{T}_n)) \sim \lambda_{k,r}.$$

Proof. We can find the expected number of k-cycles by adding together the probabilities of all possible cycles. By Lemma 3.6, sets of edges (in $\mathcal{T}_n$) with the same path structure will occur with the same probability. We can therefore group these together and sum over all $I \in \mathcal{I}_k$. Given a particular I, we can build all possible pre-cycles with this path-structure by starting with a pre-cycle on $|I|$ independent "pseudo"-edges and then replacing each pseudo-edge with a path of the correct length. By Lemma 3.16 the number of possible pre-cycles on $|I|$ independent edges is

$$\frac{\left(\frac{n}{r}\right)_{|I|}}{2|I|}(r(r-1))^{|I|},$$

we must then multiply this by the $\binom{|I|}{i_1,\ldots,i_k}$ ways of replacing the pseudo-edges by the actual paths and the $\left(\frac{n}{r}-|I|\right)_{k-|I|} r^{k-|I|}$ possible choices of vertices to make up these paths. Finally we multiply by the probability of the pre-cycle being present which, by Lemma 3.6, is

$$\frac{1}{n^k}\prod_{l=1}^{k}(l+1)^{i_l}.$$

Putting all this together gives us that

$$\begin{aligned}
\mathbf{E}X_k(R_r(\mathcal{T}_n)) &= \sum_{I\in\mathcal{I}_k}\binom{|I|}{i_1,\ldots,i_k}\frac{\left(\frac{n}{r}\right)_{|I|}}{2|I|}(r(r-1))^{|I|}\left(\frac{n}{r}-|I|\right)_{k-|I|} r^{k-|I|}\frac{1}{n^k}\prod_{l=1}^{k}(l+1)^{i_l} \\
&= \sum_{I\in\mathcal{I}_k}\binom{|I|}{i_1,\ldots,i_k}\left(\frac{n}{r}\right)_k\frac{r^k(r-1)^{|I|}}{2|I|}\frac{1}{n^k}\prod_{l=1}^{k}(l+1)^{i_l} \\
&= \lambda_{k,r}\left(\frac{n}{r}\right)_k\left(\frac{r}{n}\right)^k. \qquad (3.23)
\end{aligned}$$

We can bound $\left(\frac{n}{r}\right)_k$ above by

$$\left(\frac{n}{r}\right)^k,$$

and below by

$$\left(\frac{n}{r}-k+1\right)^k=\left(\frac{n}{r}\right)^k\left(1-\frac{(k-1)r}{n}\right)^k,$$

from these bounds and equation (3.23) we see that

$$\lambda_{k,r}\left(1-\frac{(k-1)r}{n}\right)^k \le \mathbf{E}X_k(R_r(\mathcal{T}_n)) \le \lambda_{k,r}.$$

■

From this we get that for fixed $r > 1$ the expected numbers of k-cycles for the first few k are:

$$\begin{aligned}
\mathbf{E}X_1(R_r(\mathcal{T}_n)) &\sim r-1, \\
\mathbf{E}X_2(R_r(\mathcal{T}_n)) &\sim (r-1)^2 + \frac{3}{2}(r-1), \\
\mathbf{E}X_3(R_r(\mathcal{T}_n)) &\sim \frac{4}{3}(r-1)^3 + 3(r-1)^2 + 2(r-1), \\
\mathbf{E}X_4(R_r(\mathcal{T}_n)) &\sim 2(r-1)^4 + 6(r-1)^3 + \frac{25}{4}(r-1)^2 + \frac{5}{2}(r-1).
\end{aligned}$$

The asymptotic distribution of the number of k-cycles in $R_r(\mathcal{T}_n)$

In the final part of this section we will use the method of moments to find the asymptotic distribution of $X_k(R_r(\mathcal{T}_n))$. We will prove that

$$\mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t \sim (\lambda_{k,r})^t$$

as n tends to infinity and hence conclude that $X_k(R_r(\mathcal{T}_n))$ has asymptotically a Poisson distribution with parameter $\lambda_{k,r}$.

Theorem 3.18 *Let k and r be fixed positive integers, with $r \geq 2$, and integer $n \geq r$ be such that n/r is a positive integer. Then*

$$X_k(R_r(\mathcal{T}_n)) \xrightarrow{\mathrm{D}} \mathrm{Po}(\lambda_{k,r}).$$

as n tends to infinity.

Proof. Let $A_{n,r,k}$ be the set of all r-pre-cycles of length k on the vertices of $\mathcal{T}_n$, and for any $\alpha \in A_{n,r,k}$ let $Z_{n,\alpha}$ be equal to one if all the edges of α are present in $\mathcal{T}_n$, zero otherwise. Given $\alpha \in A_{n,r,k}$, let $\hat{\alpha}$ be the cycle formed by the edges of α in the r-reduced graph. We can now rewrite $X_k(R_r(\mathcal{T}_n))$ as

$$X_k(R_r(\mathcal{T}_n)) = \sum_{\alpha \in A_{n,r,k}} Z_{n,\alpha}.$$

This means that $X_k(R_r(\mathcal{T}_n))$ meets all the conditions for equation (2.30) and so the factorial moments of $X_k(R_r(\mathcal{T}_n))$ can be expressed as

$$\begin{aligned} \mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t &= \sum_{\alpha_1,\ldots,\alpha_t}^{*} \Pr[Z_{n,\alpha_1} = \ldots = Z_{n,\alpha_t} = 1] \\ &= \sum_{\alpha_1,\ldots,\alpha_t}^{*} \Pr[\alpha_1, \ldots, \alpha_t \in \mathcal{T}_n]. \end{aligned}$$

A lower bound on this expression is obtained by considering only sets of pre-cycles that will reduce to vertex disjoint cycles in $R_r(\mathcal{T}_n)$. Given the t path structures $I_1, \ldots, I_t$ for $\alpha_1, \ldots, \alpha_t$, the number of ways to choose the sets of vertices in which each pre-cycle is formed is

$$\prod_{j=1}^{t} \left(\frac{\left(\frac{n}{r} - k(j-1)\right)_k}{2|I_j|} \right) r^k (r-1)^{|I_j|}. \tag{3.24}$$

There are also

$$\prod_{j=1}^{t} \binom{|I_j|}{i_1^j, \ldots, i_k^j} \tag{3.25}$$

choices for how to replace the pseudo-edges with paths. By Lemma 3.6, the number of trees on n vertices containing a given set of edges I_j is

$$n^{n-k-2} \prod_{l=1}^{k} (l+1)^{i_l^j},$$

hence the probability of t disjoint pre-cycles being present is

$$\Pr[\alpha_1, \ldots, \alpha_t \in \mathcal{T}_n] = \frac{1}{n^{kt}} \prod_{j=1}^{t} \prod_{l=1}^{k} (l+1)^{i_l^j}. \tag{3.26}$$

Putting together equations (3.24), (3.25) and (3.26) and summing over all $\{I_1, \ldots, I_t\}$ gives

$$\begin{aligned}
&\mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t \geq \\
&\geq \sum_{I_1,\ldots,I_t} \left(\prod_{j=1}^{t} \left(\frac{(\frac{n}{r} - k(j-1))_k}{2|I_j|} \right) r^k (r-1)^{|I_j|} \right) \left(\prod_{j=1}^{t} \binom{|I_j|}{i_1^j, \ldots, i_k^j} \right) \left(\frac{1}{n^{kt}} \prod_{j=1}^{t} \prod_{l=1}^{k} (l+1)^{i_l^j} \right) \\
&= \frac{1}{n^{kt}} \sum_{I_1,\ldots,I_t} \prod_{j=1}^{t} \left(\left(\frac{(\frac{n}{r} - k(j-1))_k}{2|I_j|} \right) r^k (r-1)^{|I_j|} \binom{|I_j|}{i_1^j, \ldots, i_k^j} \prod_{l=1}^{k} (l+1)^{i_l^j} \right),
\end{aligned} \tag{3.27}$$

and then by noticing that

$$\left(\frac{n}{r} - k(j-1) \right)_k$$

can be bounded below by

$$\left(\frac{n}{r} - k(t-1) \right)^k,$$

and taking everything that does not depend on j outside of the product we can see that (3.27) is at least

$$\begin{aligned}
&\left(\frac{r}{n} \right)^{kt} \left(\frac{n}{r} - k(t-1) \right)^{kt} \sum_{I_1,\ldots,I_t} \prod_{j=1}^{t} \left(\left(\frac{1}{2|I_j|} \right) (r-1)^{|I_j|} \binom{|I_j|}{i_1^j, \ldots, i_k^j} \prod_{l=1}^{k} (l+1)^{i_l^j} \right) \\
&= \left(1 - \frac{kr(t-1)}{n} \right)^{kt} \sum_{I_1,\ldots,I_t} \prod_{j=1}^{t} \left(\binom{|I_j|}{i_1^j, \ldots, i_k^j} \frac{(r-1)^{|I_j|}}{2|I_j|} \prod_{l=1}^{k} (l+1)^{i_l^j} \right).
\end{aligned}$$

By the multinomial theorem,

$$\sum_{I_1,\ldots,I_t}\prod_{j=1}^{t}\left(\binom{|I_j|}{i_1^j,\ldots,i_k^j}\frac{(r-1)^{|I_j|}}{2|I_j|}\prod_{l=1}^{k}(l+1)^{i_l^j}\right)=\left(\sum_{I}\binom{|I|}{i_1,\ldots,i_k}\frac{(r-1)^{|I|}}{2|I|}\prod_{l=1}^{k}(l+1)^{i_l}\right)^t$$

and thus

$$\mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t \geq \lambda_{k,r}^t\left(1-\frac{kr(t-1)}{n}\right)^{kt}.$$

To bound the expression above, we can still group the terms as

$$\sum_{\alpha_1,\ldots,\alpha_t}^{*}\Pr[\alpha_1,\ldots,\alpha_t\in\mathcal{T}_n]$$

based on the path-structure of $\alpha_1,\ldots,\alpha_t$, but of course, we need to account for the fact that the cycles may not be disjoint. Given $I_1,\ldots,I_t\in\mathcal{I}_k$, the number of ways to choose the ordered sets of vertices on which each pre-cycle is formed can be bounded above by

$$\prod_{j=1}^{t}\left(\frac{\left(\frac{n}{r}\right)_k}{2|I_j|}\right)r^k(r-1)^{|I_j|}, \tag{3.28}$$

(note that this overcounts by including cases where edges from the various pre-cycles form a cycle on the vertices of $\mathcal{T}_n$ as in Figure 3.7).

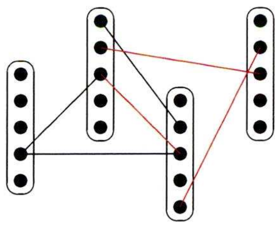

Figure 3.7: The red and black edges form two 3-cycles in the reduced graph. Individually, neither set forms a cycle in $\mathcal{T}_n$, but taken together there is a cycle on the leftmost three empires consisting of two black edges and one red.

When non-vertex disjoint cycles are allowed, it is possible that some pre-cycles will share edges. We will now give a weak upper bound on the expected number of sets of t pre-cycles within $\mathcal{T}_n$ where $l > 1$ of the edges appear in multiple pre-cycles, and show that this is small compared to $\lambda_{k,r}^t$. If l edges are shared between the pre-cycles then there are a total of $kt - 2l$ vertices making up the pre-cycles. The number of ways to choose the sets of vertices on which each pre-cycle is formed is thus at most

$$Cn^{kt-2l},$$

for some constant C. By Lemma 3.6, the probability that $\mathcal{T}_n$ contains a given set of $kt - l$ edges can be bounded above by

$$\frac{2^{kt-l}}{n^{kt-l}},$$

hence the expected number of such sets of pre-cycles is at most

$$\frac{2^{kt-l}C}{n^l} \leq \frac{C}{n}, \tag{3.29}$$

for some positive constant C.

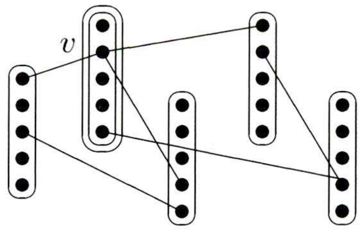

Figure 3.8: A vertex v participating in two different triangles in a 5-reduced graph.

We next look at the probability that $R_r(\mathcal{T}_n)$ contains the cycles $\hat{\alpha}_1, \ldots, \hat{\alpha}_t$ such that all pre-cycles are edge disjoint in $\mathcal{T}_n$. Such pre-cycles may also be vertex disjoint, or there may be vertices that appear in more than one pre-cycle. We will show that the probability of a given set of pre-cycles appearing in the tree (as given by Lemma 3.6) is maximised when no vertex appears in more than one pre-cycle.

Suppose that one vertex v of $\mathcal{T}_n$ appears in m different pre-cycles (say $\alpha_1, \ldots, \alpha_m$, without loss of generality), such that for any $1 \leq i \leq m$, the length of the path in α_i containing v is l_i. Figure 3.8 gives an example for $m = 2$, and $k = 3$ (for the given example $l_1 = 2$, and $l_2 = 3$, note that the rightmost cycle contains a second component which has a vertex within the empire containing v, but this is not counted as it does not contain v itself). These paths form a connected component in $\mathcal{T}_n$ of size

$$\left(\sum_{i=1}^{m} l_i\right) + 1$$

and hence by Lemma 3.6, the probability that $\mathcal{T}_n$ contains all the edges of $\alpha_1, \ldots, \alpha_t$ is equal to the probability that it contains all of the paths that do not contain v multiplied by

$$\frac{1}{n^{\sum_{i=1}^{m} l_i}} \left(1 + \sum_{i=1}^{m} l_i\right).$$

Next, consider a graph $\mathcal{T}_n'$ that is identical to $\mathcal{T}_n$ except that instead of all meeting at vertex v $\alpha_1, \ldots, \alpha_m$ all contain different vertices in the same empire (Figure 3.9 gives a simple example for the graph in Figure 3.8). Now, for each $1 \leq i \leq m$ there is a connected component of size $l_i + 1$ that was in the shared component of $\mathcal{T}_n$. Therefore by Lemma 3.6, the probability that $\mathcal{T}_n'$ contains all the edges of $\alpha_1, \ldots, \alpha_t$ is equal to the probability that it

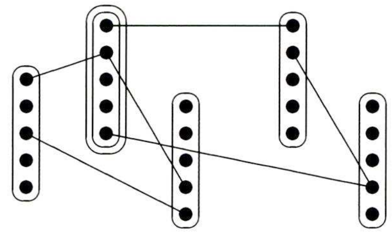

Figure 3.9: An empire participating in two different triangles in a 5-reduced graph. Note that no vertex appears in more than one triangle.

contains all of the other paths multiplied by

$$\frac{1}{n^{\sum_{i=1}^{m} l_i}} \prod_{i=1}^{m} (l_i + 1).$$

The probability is at its highest when all of the paths are vertex disjoint, hence equation (3.26) can be used as an upper bound on probability.

We can now obtain an upper bound on the t^{th} factorial moment of $X_k(R_r(\mathcal{T}_n))$ by putting together equations (3.25), (3.26), (3.28) and (3.29), and summing over all $\{I_1, \dots, I_t\}$

$$\begin{aligned}
&\mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t \leq \\
&\leq \sum_{I_1,\dots,I_t} \left(\prod_{j=1}^{t} \binom{\left(\frac{n}{r}\right)_k}{2|I_j|} r^k (r-1)^{|I_j|} \right) \left(\prod_{j=1}^{t} \binom{|I_j|}{i_1^j, \dots, i_k^j} \right) \left(\frac{1}{n^{kt}} \prod_{j=1}^{t} \prod_{l=1}^{k} (l+1)^{i_l^j} \right) + \frac{C}{n} \\
&= \frac{1}{n^{kt}} \sum_{I_1,\dots,I_t} \prod_{j=1}^{t} \left(\binom{\left(\frac{n}{r}\right)_k}{2|I_j|} r^k (r-1)^{|I_j|} \binom{|I_j|}{i_1^j, \dots, i_k^j} \prod_{l=1}^{k} (l+1)^{i_l^j} \right) + \frac{C}{n} \\
&\leq \sum_{I_1,\dots,I_t} \prod_{j=1}^{t} \left(\binom{|I_j|}{i_1^j, \dots, i_k^j} \frac{(r-1)^{|I_j|}}{2|I_j|} \prod_{l=1}^{k} (l+1)^{i_l^j} \right) + \frac{C}{n},
\end{aligned}$$

and thus, invoking the multinomial theorem again,

$$\mathbf{E}(X_k(R_r(\mathcal{T}_n)))_t \leq \lambda_{k,r}^t + \frac{C}{n}.$$

The overall result now follows by Theorem 2.8. ■

3.4.5 Cliques

Let K_s be a clique of size s, i.e. a complete subgraph on s vertices. As with cycles, a tree will by definition contain no instances of K_s for $s > 2$, however once again it is possible for the reduced graph $R_r(T_n)$ to contain higher cliques as a result of paths within T_n. Given a set of s distinct vertices $\beta = \{v^{(1)}, \ldots, v^{(s)}\} \in R_r(T_n)$, there is a clique in $R_r(T_n)$ on β if for every pair of empires $(B_{v^{(i)}}, B_{v^{(j)}})$ with $i \neq j$ there exists at least one edge connecting a vertex in the set $\{v_1^{(i)}, \ldots, v_r^{(i)}\}$ to a vertex in $\{v_1^{(j)}, \ldots, v_r^{(j)}\}$.

Let $Y_s(R_r(\mathcal{T}_n))$ be the number of s-cliques in $R_r(\mathcal{T}_n)$. We already know that

$$Y_3(R_r(\mathcal{T}_n)) \xrightarrow{\mathrm{D}} \mathrm{Po}\left(\frac{4r^3}{3} - r^2 + \frac{1}{3}\right).$$

(this is true because $Y_3(R_r(\mathcal{T}_n)) \equiv X_3(R_r(\mathcal{T}_n)))$. The natural question is what happens for $k > 3$. The next results show that large cliques in $R_r(\mathcal{T}_n)$ are quite rare. Lemma 3.19 uses a simple counting argument to prove that no r-reduced graph contains a copy of K_{2r+1}. The subsequent Theorem, via an elementary union bound, shows that, in fact, even K_4 is rare in the r-reduced graph of a random tree, as long as r is a fixed positive number. As a corollary we obtain a full characterisation of the size of the largest cliques in $R_r(\mathcal{T}_n)$, for any fixed value of $r \geq 1$. In that result $\omega(G)$ is the size of the largest clique in the graph G.

Lemma 3.19 *For any positive integers r and n, with $n \geq r$ and such that n/r is a positive integer, and any labelled tree T_n, the graph $R_r(T_n)$ does not contain any clique of size greater than $2r$.*

Proof. For any positive integer s, any induced subgraph of T_n consisting of s empires (rs vertices) is itself an empire forest and as such has at most

$rs - 1$ edges, a clique on s vertices has one edge between each pair of vertices, or

$$\frac{s(s-1)}{2}$$

edges in total. A subgraph consisting of $2r$ empires may therefore have at most $2r^2 - 1$ edges, while a K_{2r} clique has

$$r(2r-1) = 2r^2 - r \leq 2r^2 - 1$$

edges. A subgraph on $2r+1$ empires however, may have at most $2r^2 + r - 1$ edges while a clique on the corresponding vertices in $R_r(\mathcal{T}_n)$ would need

$$r(2r+1) = 2r^2 + r > 2r^2 + r - 1$$

edges. Since the number of edges required for the clique is more than may be present on the subgraph, no clique of size greater than $2r$ may exist in $R_r(T_n)$. ■

Lemma 3.19 is valid for any tree T_n, and any value of r, even depending on n. Also, note that the study of the size of the largest cliques in $R_r(T_n)$ is strictly related to the analysis of the chromatic properties of $R_r(\mathcal{T}_n)$ as the size of the largest clique in a graph is a natural lower bound on its chromatic number. In fact Theorem 4.1 in Chapter 4 provides an alternative proof of Lemma 3.19. The next result shows that, in fact, much smaller cliques are rare in the r-reduced graph of a random tree.

Theorem 3.20 *For any fixed positive integer r, and integer $n \geq r$ such that n/r is a positive integer, $R_r(\mathcal{T}_n)$ a.a.s. contains no clique of size at least four.*

Proof. The result is obvious for $r = 1$. For $r \geq 2$, the number of ways in which a set of four empires can be chosen from the n/r empires in $R_r(\mathcal{T}_n)$ is

$$\binom{n/r}{4} < \left(\frac{n}{r}\right)^4 \frac{1}{4!},$$

for each set of empires there are at most r^{12} possible sets of six edges on the vertices of these empires in $\mathcal{T}_n$ that will lead to a clique on the empires in $R_r(\mathcal{T}_n)$. From this we have that there are at most

$$\left(\frac{n}{r}\right)^4 \frac{r^{12}}{4!} \tag{3.30}$$

sets of six edges in $\mathcal{T}_n$ corresponding to 4-cliques in $R_r(\mathcal{T}_n)$.

By Lemma 3.6, the number of trees on n vertices in which a given set of six edges is present is maximised when no two edges share an end point, in which case there are

$$64n^{n-8} \tag{3.31}$$

trees containing these edges. By multiplying (3.30) and (3.31) and dividing by the total number of trees on n vertices we can obtain an upper bound on the probability of the reduced graph of a random tree containing at least one K_4

$$\begin{aligned}
\Pr[K_4 \in R_r(\mathcal{T}_n)] &\leq 64n^{n-8} \left(\frac{n}{r}\right)^4 \frac{r^{12}}{4!} \frac{1}{n^{n-2}} \\
&= \frac{8r^8}{3n^2} \\
&= O\left(\frac{1}{n^2}\right).
\end{aligned}$$

As any larger clique must contain a K_4 within it, this also serves as an upper bound on the probability of $R_r(\mathcal{T}_n)$ containing any clique of size at least 4.

■

Corollary 3.21 *Let r be a fixed positive integer, and integer $n \geq r$ be such that n/r is a positive integer. Then*

$$\Pr[\omega(R_r(\mathcal{T}_n)) = 2] \sim \mathrm{e}^{-\lambda_{3,r}} \qquad \textit{and} \qquad \Pr[\omega(R_r(\mathcal{T}_n)) = 3] \sim 1 - \mathrm{e}^{-\lambda_{3,r}}$$

as n tends to infinity.

Proof. By Theorem 3.20 $Pr[\omega(R_r(\mathcal{T}_n)) > 3]$ tends to zero as n grows to infinity. The event "$\omega(R_r(\mathcal{T}_n)) = 3$" is equivalent to "$X_3(R_r(\mathcal{T}_n)) > 0$". By the results in Section 3.4.4

$$\Pr[\omega(R_r(\mathcal{T}_n)) = 3] \to 1 - \mathrm{e}^{-\lambda_{3,r}}.$$

Finally, clearly, since each graph $R_r(\mathcal{T}_n)$ contains at least one edge (for $n \geq 2$)

$$\Pr[\omega(R_r(\mathcal{T}_n)) = 2] = 1 - \Pr[\omega(R_r(\mathcal{T}_n)) \geq 3].$$

■

The (weak) consequence of the analysis in this section on the empire colourability of random trees is that, for fixed values of $r \geq 2$, there is, asymptotically, a constant probability that $\mathcal{T}_n$ cannot be coloured with just two colours (because it contains a triangle). In Chapter 5 we will improve on this by showing that, in fact, for each $r \geq 2$ there exists a value $s_r \geq 2$ such that a random tree on n vertices $\mathrm{C}(\mathcal{T}_n, r, s)$ is empty a.a.s. for any $s \leq s_r$.

3.5 Comparisons with Other Types of Random Graphs

In this section we will look at how the structural properties of the reduced graphs of random trees compare to those of other types of random graph, primarily random regular graphs, random planar graphs, and the Erdős-Rényi random graphs with a fixed number of edges. Since the number of vertices and edges is fixed in $R_r(\mathcal{T}_n)$, we will consider random graphs on n/r vertices, which are either regular graphs of degree $2r$ (denoted by $\mathcal{G}_{\frac{n}{r},2r}$), or uniformly distributed over the set of all graphs on n/r vertices with $n-1$ edges (denoted by $G(\frac{n}{r}, n-1)$ or, when interested in planar graphs, by $P(\frac{n}{r}, n-1)$). A planar graph on m vertices can have at most $3m-6$ edges and so obviously no comparable graphs can be considered for $r > 3$, however as n grows large, graphs on $\frac{n}{3}$ vertices with $n-6$ edges provide a reasonable comparison point for $r = 3$. Note that such graphs are maximally planar (see [26, Chapter 4]) and as such they are triangulations.

In this section we deal with the properties considered in this chapter. We defer a comparison of the chromatic properties of all these models to the discussion at the end of Chapter 5.

3.5.1 Connectivity

$R_r(\mathcal{T}_n)$ must always be connected since the tree from which it was reduced is also connected, however this is not necessarily the case with other types of random graphs. In particular Erdős-Rényi graphs with a relatively low number of edges are quite likely to be disconnected [16].

In the case of random regular graphs it has been shown in [14] that any random regular graph of degree $d \geq 3$ will a.a.s. be d-connected. This is a stronger claim than that given for $R_r(\mathcal{T}_n)$: in Theorem 3.12 it was proved that $R_r(\mathcal{T}_n)$ is a.a.s. $(r-1)$-connected.

Looking now at random planar graphs, it was shown in [53] that a random planar graph with no restriction on the number of edges has at least a probability of e^{-1} of being connected. It was conjectured in the same paper that the true probability is closer to 0.95. When we fix the number of edges, a planar graph with $\frac{n}{3}$ vertices and $n-6$ edges (i.e. a graph having a density similar to that of $R_3(T_n)$, for any T_n) must be connected, since if the graph was disconnected it would be possible to add an edge joining two components (which is not possible in a triangulation. For completeness, we mention that Gerke *et al.* [35] show that for any random planar graph on m vertices with fixed number of edges qm with $q < 3$ there is always at least a constant probability of the graph being disconnected, furthermore there is at least a constant probability that the graph will contain an isolated vertex.

3.5.2 Degree Sequence

By definition all vertices of a regular graph have degree d, thus questions about degree sequence are trivial for this kind of graph. For other types of random graphs we can compare results about the degree with those found for $R_r(\mathcal{T}_n)$.

Theorem 3.5 gives us that for any constant $\epsilon > 0$,

$$\frac{(1-\epsilon)\log n}{\log\log n} \leq \Delta(R_r(\mathcal{T}_n)) \leq \frac{(1+\epsilon)\log n}{\log\log n} \qquad \text{a.a.s.}$$

This is very similar to what can be proved for Erdős-Rényi random graphs with a constant average degree. Such graphs a.a.s. have maximum degree

$$\Theta\left(\frac{\log n}{\log\log n}\right)$$

(this can be obtained from results on the related, so called, $G(n,p)$ model [15, Theorem 3.7, p.66]). Similarly, random planar graphs have maximum degree

$$\Omega\left(\frac{\log n}{\log\log n}\right)$$

with probability $1 - o(1)$ [53]. Note however that, by a result of Gao and Wormald [33],

$$\Delta(P(m, 3m-6)) \sim \frac{\log m}{\log 4/3} - \frac{\log\log m}{2\log 4/3}$$

as m tends to infinity. Thus the maximum vertex degree in a random planar triangulation is somewhat larger than that of $R_3(\mathcal{T}_n)$.

A lot is known about the degree sequence of the Erdős-Rényi random graphs. For each fixed d, the probability that a given vertex in a random graph with average degree $2r$ has degree d is $(1+o(1))\frac{(2r)^d}{d!\ e^{2r}}$, as n tends to infinity. (Achlioptas and Moore, for instance, refer to this as a folklore result [3]). This is very qualitatively similar to the results on the degree distribution of $R_r(\mathcal{T}_n)$ presented in this thesis (see Lemma 3.2 earlier on in this chapter). Of course no vertex in $R_r(\mathcal{T}_n)$ can have degree less than r. Thus, results on the minimum degree in $R_r(\mathcal{T}_n)$ are very different from those relative to $G(\frac{n}{r}, n-1)$. In particular, any of the classical random graphs considered in this section (except random regular graphs) will contain (many) isolated vertices a.a.s.

3.5.3 Cycles and Cliques

It has been shown in [68], that a random regular graph will a.a.s. contain no copy of any given subgraph with more edges than vertices, hence as with $R_r(\mathcal{T}_n)$ there will be a.a.s. no cliques of size four or larger. Also as with $R_r(T_n)$, for any tree T_n, any regular graph with degree $2r$ will contain no copy of K_{2r+2} since this would require each vertex in the clique to have degree at least $2r+1$. It is however possible, albeit unlikely, for a random regular graph of degree $2r$ to contain a copy of K_{2r+1}. Regarding cycles, Bollobás showed in [13] that the numbers of short cycles of length i in a random regular graph of degree d are independent random variables tending to a Poisson distribution with mean

$$\lambda_i = \frac{(d-1)^i}{2i}.$$

This is quite close to the distribution for $R_r(\mathcal{T}_n)$ given by Theorem 3.18. The averages in that case, for the first few values of i are as follows:

$$\begin{aligned}
\lambda_{1,r} &= r-1, \\
\lambda_{2,r} &= r^2 - \frac{r+1}{2}, \\
\lambda_{3,r} &= \frac{4r^3}{3} - r^2 + \frac{1}{3}, \\
\lambda_{4,r} &= 2r^4 - 2r^3 + \frac{r^2-1}{4}
\end{aligned}$$

Similar results hold in $G(\frac{n}{r}, n-1)$. Since cycles of length i, for any $i \geq 3$, are strictly balanced graphs, and their automorphism group (the so called *dihedral group*) has order $2i$, results from [15, Chapter 4] suggest that the number of copies of the cycle of length i in $G(\frac{n}{r}, n-1)$ has asymptotically Poisson distribution with parameter $\frac{(2r)^i}{2i}$. In fact the same argument entails that the number of complete graphs on i vertices in $G(\frac{n}{r}, n-1)$ has asymptotically Poisson distribution with parameter $\frac{(2r)^i}{i!}$ (the $i!$ comes from the fact that the automorphism group of the complete graph on i vertices is the set of all permutations on $\{1, \ldots, i\}$). Thus, for large values of n, $G(\frac{n}{r}, n-1)$ will, for instance, contain a copy of K_4 with probability approximately equal to $1 - \exp\left(-\frac{2r^4}{3}\right)$ whereas, by Theorem 3.20, the probability that $R_r(\mathcal{T}_n)$ contains K_4 is negligible.

Gerke *et al.* [35] prove that for random planar graphs with fixed number of edges greater than the number of vertices, for any connected planar subgraph H there exists some constant $\alpha > 0$ such that the probability of the number of vertex disjoint copies of H being less than αn is at most $\mathrm{e}^{-\Omega(n)}$. Hence there will be a large number of short cycles and K_4 cliques. Larger cliques however will not be present since K_5 is not planar.

Chapter 4

The Empire Colouring Problem on Trees

In this chapter we will consider the empire colouring problem on trees. As explained in Section 3.1, for any $r \geq 1$, finding a proper empire colouring of a planar graph G is equivalent to finding a proper colouring of its r-reduced graph $R_r(G)$. Here, we are interested in estimating the chromatic number of $R_r(T)$, where T is a tree on n vertices.

4.1 Arbitrary Graphs

Before considering random graphs, we look at the number of colours required to give a proper colouring of the r-reduced graph of any tree. We bound the number of colours required above by using a method similar to that used by Percy John Heawood to show that all planar graphs are six-colourable [43]. This number is then shown to be necessary to properly colour the r-reduced graphs of all trees through an inductive method to create trees requiring this many colours.

Theorem 4.1 *Let r and n be positive integers such that $n \geq r$ and n/r is a positive integer. Let T_n be a tree on n vertices. Then*

$$\chi(R_r(T_n)) \leq 2r.$$

Proof. The proof is constructive. We will prove that an obvious modification of Heawood's heuristic (described at the end of Section 3.1) colours $R_r(T_n)$ using no more than $2r$ colours.

As explained in subsection 3.4.1, the average degree of the r-reduced graph of any tree is strictly less than $2r$. Due to this, there must always be at least one empire in the graph with degree at most $2r - 1$. If we select one such empire and remove it from the graph, the induced graph resulting from this is the reduced graph of a forest with $n - r$ vertices. By the same argument, this graph must also contain at least one empire of degree at most $2r - 1$ which can be removed. The process can be continued in this way with the graph being decomposed by removing one empire at a time until none are left.

With this done the graph can be built back up by adding the empires to the graph in the reverse order from how they were removed. Each empire is coloured as it is added using a simple greedy algorithm whereby the empire is assigned the first colour not used by any of its neighbours. As the maximum degree that any empire may have at the time it is added is $2r - 1$, at most $2r$ colours will be required to colour the vertex and its neighbours. ■

Note that, in contrast with what happens in the context of arbitrary planar graphs, for $r = 1$, the algorithm returns an optimal colouring of any tree (of course finding an optimal colouring of a planar graph is NP-hard [34]).

Theorem 4.2 *Let r and n be positive integers such that $n \geq r$ and n/r is a* *positive integer. There is a family of trees* $(T^r)_{r\geq 1}$ *such that*

$$\chi(R_r(T^r)) = 2r$$

for each $r \geq 1$.

Proof. We define a family of trees $(T^r)_{r\geq 1}$ such that, for all integers $r \geq 1$, the reduced graph of T^r is K_{2r}, the complete graph on $2r$ vertices. The tree T^r will have $v(r) = 2r^2$ vertices of which $l(r) = r^2 - r + 2$ have degree one. Furthermore, if C_1 and C_2 are two special vertices called the *centres* of T^r then there will be exactly r vertices of degree one, belonging to empires $1, 3, \ldots, 2r-1$ at distance $2(r-1)$ from C_1 and r vertices of degree one, belonging to empires $2, 4, \ldots, 2r$ at distance $2(r-1)$ from C_2. These sets of vertices are called Far_1 and Far_2 respectively.

The empire tree $T^1 \equiv K_2$. Assume that T^{r-1} is given consisting of empires of size $r-1$, labelled from one to $2(r-1)$, which satisfies all properties above. Add $r-1$ new vertices belonging to empire $2r-1$ and $r-1$ vertices belonging to empire $2r$. Connect each of the new vertices in empire $2r-1$ (resp. $2r$) with a distinct element of Far_1 (resp. Far_2). By adding one more vertex to each empire we can change this so that any two empires are adjacent. Choose one vertex from empire $2r-1$ (resp. $2r$), and attach r new leaves belonging to empires $2, 4, \ldots, 2r$ (resp. $1, 3, \ldots, 2r-1$) to this vertex. The resulting tree is an r-empire tree. It has $l(r) = l(r-1) + 2(r-1)$ vertices of degree one and $v(r) = v(r-1) + 2(r-1) + 2r$ vertices in total and reduces to K_{2r}.

■

Remarks. The results in this section seem to solve the question of the empire colourability of trees - the lower bound on the number of colours

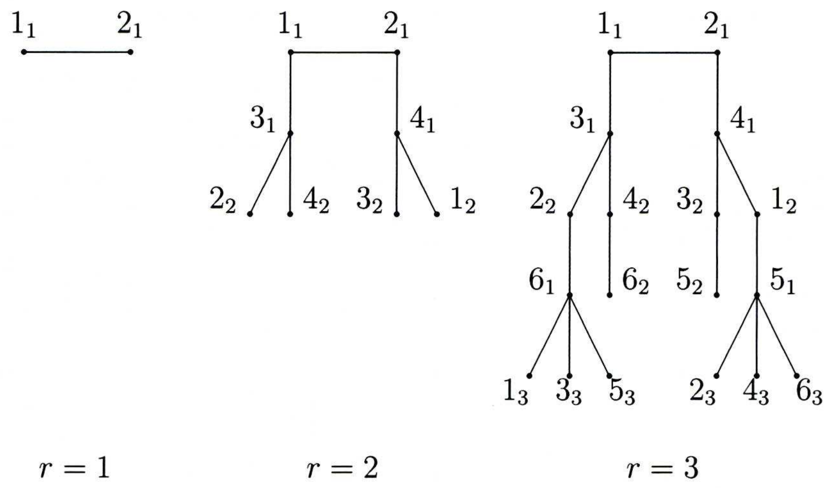

Figure 4.1: The trees T^1, T^2, and T^3 built following the method given in Theorem 4.2. In each tree, the vertex labelled i_j is the j^{th} element of empire i. Note that there is at least one edge between any pair of empires in each tree and hence each reduces to a clique K_{2r}.

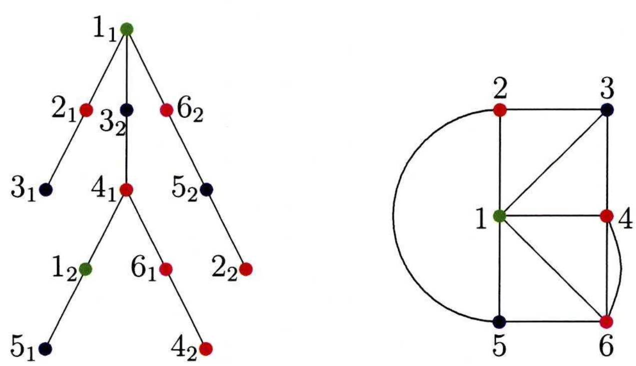

Figure 4.2: An example of a tree with $r = 2$ that does not reduce to a graph containing K_{2r} and yet requires $2r$ colours for a proper colouring.

required is equal to the upper bound - however, this is not the whole story. The trees defined in Theorem 4.2 all reduce to K_{2r}, yet by Theorem 3.20 in Chapter 3 the reduced graph of a random tree will a.a.s. contain no cliques of size greater than three. Of course, the presence of a large clique is not a necessary condition for a graph to require $2r$ colours - see for example figure 4.2 which has $r = 2$ and chromatic number 4, but this small graph contains six triangles when Theorem 3.17 predicted that even large graphs would have only about $\frac{17}{3}$ and many edges are in two different triangles, all of which makes it unlikely to appear as a subgraph of $R_r(\mathcal{T}_n)$ for large n.

The obvious question that arises from this is as follows: "what number of colours is required to give a proper r-empire colouring of almost all trees?" Is it really the case that $2r$-chromatic trees are sufficiently common that this worst case will inevitably come up and the full $2r$ colours will be required, or taking the other extreme will the relative sparseness of the graphs with no large cliques and few triangles mean that most are three-colourable (It is known [39] that all planar graphs containing no triangles are three-colourable)? Or does the true answer lie somewhere in between, possibly depending on the value of r?

4.2 Colouring Algorithms

Before addressing the questions above from the asymptotic point of view, it is instructive to investigate them empirically. In this section we provide evidence supporting the following claims:

1. The value $2r$ is often a very crude upper bound on $\chi(R_r(T))$.

2. There exist algorithms that return colourings using as few as at most $r + 3$ colours for most trees.

The claims will be justified by providing an empirical evaluation of the performances of a number of simple heuristics for solving the given colouring problem. We first look at a variant of Heawood's strategy, based on peeling off vertices of minimum degree. Then we will describe a revised version of such minimum degree heuristic, inspired by the well-known Brooks' algorithm [19]. Finally we will consider a simple list colouring strategy [49].

The final section of the chapter will investigate the possibility to devise even better algorithms by pairing up the heuristics considered in this section with a simple pre-processing strategy which has had applications in the context of colouring random regular graphs [64, 65].

Experiment Design. With each of the algorithms described in below we run the following experiment:

> We take in turn $r = 2, 3, 4, 5, 6, 10$, and we run the algorithm on the reduced graph of 400 random trees with n vertices, for values of n equal to 120, 600, 1200, and then for $n = 3000 * i$ for $i \in \{1, \ldots, 20\}$. In each case we record the full distribution of the number of graphs coloured with s colours for values of s up to $2r$.

In fact the tables in the following pages only show the distributions for $s > s_r$, where s_r is the almost sure lower bound on $\chi(R_r(\mathcal{T}_n))$ proved in Theorem 5.1 of Chapter 5. The values of s_r for the values of r considered in this chapter are reported in Table 4.1.

Figure 4.3 at the end of the section helps in comparing the performances of the three heuristics considered.

r	2	3	4	5	6	10
s_r	2	3	3	4	4	6

Table 4.1: A.a.s. lower bounds on the chromatic number of $R_r(\mathcal{T}_n)$ for different values of r.

4.2.1 Heawood Colouring

One of the simplest methods for finding an empire colouring of a tree is by colouring its reduced graph using the greedy heuristic that is implicit in the proof of Theorem 4.1. The graph is decomposed one empire at a time by removing, each time, an empire with degree less than $2r$. In fact at each step our algorithm chooses a vertex of minimum degree. This process is repeated until we are left with a single empire which is given colour 1. The graph is then built up in the reverse order to which it was decomposed, with each empire being coloured as it is added with the first colour not used by any of its neighbours. In the following algorithm, the array Order lists the vertices in the order in which they are to be coloured and the i^{th} element in the array Colour is the colour of vertex i. This algorithm will of course never use more than $2r$ colours but it may use less, especially if most of the empires have close to the minimum degree r when they are added to Order. The following pseudo-code describes the strategy at hand.

Algorithm Heawood($\mathcal{H}$)

Set $G = \mathcal{H}$. Set $m = |V(\mathcal{H})|$.

for $i = 0$ to $m - 2$ **do**

Set v = vertex of minimum degree in G.

Set Order$[m - i] = v$.

Set $G = G - v$.

end for

Set u = remaining vertex in G.

Set Colour[u] = 1.

for $i = 2$ to m **do**

Set v = Order[i].

Set $G = G + v$, adding an edge between v and any vertex of G if this edge is present in $\mathcal{H}$.

Set Colour[v] = First colour that is not used by any neighbour of v.

end for

Return coloured graph.

$n\backslash r$	2	3	4	5	6	10
120	(72,328)	(184,216,0)	(16,295,88,1,0)	(120,257,22,1,0,0)	(45,236,119,0,0,0,0,0)	(221,91,13,0,0,0,...)
600	(0,400)	(6,394,0)	(0,88,308,4,0)	(0,209,191,0,0,0)	(0,20,314,66,0,0,0,0)	(1,64,240,91,4,0,...)
1200	(0,400)	(0,400,0)	(0,19,381,0,0)	(0,123,276,1,0,0)	(0,0,288,112,0,0,0,0)	(0,1,186,203,10,0,...)
3000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,21,370,9,0,0)	(0,0,183,217,0,0,0,0)	(0,0,37,322,41,0,...)
6000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,393,7,0,0)	(0,0,78,322,0,0,0,0)	(0,0,1,283,116,0,...)
9000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,384,16,0,0)	(0,0,21,377,2,0,0,0)	(0,0,0,291,109,0,...)
12000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,373,27,0,0)	(0,0,9,389,2,0,0,0)	(0,0,0,207,192,1,0,...)
15000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,375,25,0,0)	(0,0,3,393,4,0,0,0)	(0,0,0,157,240,3,0,...)
18000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,368,32,0,0)	(0,0,1,390,9,0,0,0)	(0,0,0,140,260,0,...)
21000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,361,39,0,0)	(0,0,1,390,9,0,0,0)	(0,0,0,108,292,0,...)
24000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,351,49,0,0)	(0,0,0,392,8,0,0,0)	(0,0,0,100,299,1,0,...)
27000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,350,50,0,0)	(0,0,0,387,13,0,0,0)	(0,0,0,69,330,1,0,...)
30000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,353,47,0,0)	(0,0,0,384,16,0,0,0)	(0,0,0,56,343,1,0,...)
33000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,341,59,0,0)	(0,0,0,396,4,0,0,0)	(0,0,0,47,349,4,0,...)
36000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,327,73,0,0)	(0,0,0,384,16,0,0,0)	(0,0,0,48,349,3,0,...)
39000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,334,66,0,0)	(0,0,0,384,16,0,0,0)	(0,0,0,42,355,3,0,...)
42000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,332,68,0,0)	(0,0,0,385,15,0,0,0)	(0,0,0,25,371,4,0,...)
45000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,333,67,0,0)	(0,0,0,391,9,0,0,0)	(0,0,0,21,376,3,0,...)
48000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,323,77,0,0)	(0,0,0,390,10,0,0,0)	(0,0,0,16,382,2,0,...)
51000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,310,90,0,0)	(0,0,0,383,17,0,0,0)	(0,0,0,10,382,8,0,...)
54000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,304,96,0,0)	(0,0,0,374,26,0,0,0)	(0,0,0,15,373,12,0,...)
57000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,306,94,0,0)	(0,0,0,378,22,0,0,0)	(0,0,0,6,391,3,0,...)
60000	(0,400)	(0,400,0)	(0,0,400,0,0)	(0,0,319,81,0,0)	(0,0,0,376,24,0,0,0)	(0,0,0,6,389,5,0,...)

Table 4.2: The number of colours used by the Heawood colouring algorithm. Note that, by Theorem 5.1 (see also Table 4.1) for $r = 2$ a.a.s. either three or four colours are needed, for $r \in \{3, 4\}$ we need at least four (and at most $2r$), for $r \in \{5, 6\}$ we need at least five colours, and for $r = 10$, the minimum value is seven.

Table 4.2 gives the number of colours used by the Heawood colouring algorithm to colour (the r-reduced graphs of) random trees with n in the range $120, \ldots, 60000$ and r in the set $\{2, \ldots, 6, 10\}$. Even for graphs on only a few hundred vertices, the algorithm almost always uses the full $2r$ colours for $r = 2$. However, for larger r, the number of colours used seems to level out to values that are never larger than $r + 2$ (see the values in Table 4.2 or the averages in the top diagram of Figure 4.3). We speculate that this may be due to fact that during the process the minimum degree in the graph G is always very small. Empirical evidence suggests that initially this quantity is r, then it briefly goes up to $r+2$ or $r+3$ but then plunges down more and more often to ever smaller values (even one towards the end of the process).

One obvious modification to the algorithm is that instead of colouring each vertex with the first colour that is not used by its neighbours, we instead use the unused colour that is most common in the graph so far. This however does not seem to have any effect beyond changing the distribution of colours so that instead of lower numbered colours being used more often, the colours that were used a lot early on become the most common overall.

4.2.2 Brooks Colouring

Brooks' Theorem [19] states that for any connected graph G that is not a clique or an odd cycle, the chromatic number $\chi(G)$ is at most the maximum degree of the graph $\Delta(G)$. This in itself is not particularly useful for the graphs we are considering since by Theorem 3.5, almost all trees reduce to graphs with maximum degree

$$\frac{\log n}{\log \log n}(1 + o(1)),$$

which is far too large. However, a method used in one of the proofs of such result (see for example [26, Chapter 5, pages 99–100]) provides a possible way to improve Heawood's heuristic described in the last section.

As before, given a graph, we start by peeling off minimum degree vertices until a single vertex is left. Then we put the vertices back in in reverse order. Suppose that at an arbitrary stage of this process we have coloured all vertices with some s colours. Suppose that we want to colour some empire v and that, around v, we already have vertices coloured with all s distinct colours used so far. We might solve the problem by increasing the number of available colours (this is what is done in Heawood's algorithm). Alternatively, for each pair of colours i and j we may, as a last resort, look at the induced subgraph $H_{i,j}$ consisting of all empires of $G - v$ coloured i or j. If the neighbours of v coloured i and j lie in separate components of $H_{i,j}$ then it is possible to switch the colours in one of these components so that v now has two neighbours of the same colour and one colour is left free. If this fails we will have to pick a brand new colour, and the process will revert to mimic Heawood's heuristic.

The results of the standard experiment for this refined heuristic are shown in Table 4.3. The algorithm performs better than the Heawood algorithm for all tested values of n and r. The magnitude of the improvements, and the fact that, in some cases, they seem to be decreasing with the size of the graphs considered prevents us from making any strong claim on this algorithm. Nevertheless it seems a reasonable heuristic, at least for small graphs.

4.2.3 List Colouring

In this section we present the greedy heuristic that seems to give the best results. The algorithm List($\mathcal{H}$) takes as input a graph $\mathcal{H}$ that may already be partially coloured.

Algorithm List($\mathcal{H}$)

Set $m = |V(\mathcal{H})|$.

$n \backslash r$	2	3	4	5	6	10
120	(369,31)	(397,3,0)	(190,210,0,0,0)	(368,16,0,0,0,0)	(224,175,1,0,0,0,0,0)	(202,83,19,0,0,...)
600	(310,90)	(398,2,0)	(2,398,0,0,0)	(306,94,0,0,0,0)	(0,399,1,0,0,0,0,0)	(67,329,4,0,0,...)
1200	(213,187)	(397,3,0)	(0,400,0,0,0)	(166,234,0,0,0,0)	(0,395,5,0,0,0,0,0)	(0,378,22,0,0,...)
3000	(61,339)	(380,20,0)	(0,400,0,0,0)	(9,391,0,0,0,0)	(0,371,29,0,0,0,0,0)	(0,256,144,0,0,...)
6000	(3,397)	(338,62,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,284,116,0,0,0,0,0)	(0,75,325,0,0,...)
9000	(0,400)	(249,151,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,184,216,0,0,0,0,0)	(0,14,386,0,0,...)
12000	(0,400)	(202,198,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,116,284,0,0,0,0,0)	(0,2,398,0,0,...)
15000	(0,400)	(130,270,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,52,348,0,0,0,0,0)	(0,0,400,0,0,...)
18000	(0,400)	(71,329,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,32,368,0,0,0,0,0)	(0,0,400,0,0,...)
21000	(0,400)	(61,339,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,15,385,0,0,0,0,0)	(0,0,400,0,0,...)
24000	(0,400)	(34,366,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,14,386,0,0,0,0,0)	(0,0,400,0,0,...)
27000	(0,400)	(29,371,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,6,394,0,0,0,0,0)	(0,0,394,6,0,...)
30000	(0,400)	(9,391,0)	(0,399,1,0,0)	(0,400,0,0,0,0)	(0,3,397,0,0,0,0,0)	(0,0,398,2,0,...)
33000	(0,400)	(7,393,0)	(0,399,1,0,0)	(0,400,0,0,0,0)	(0,3,397,0,0,0,0,0)	(0,0,395,5,0,...)
36000	(0,400)	(4,396,0)	(0,400,0,0,0)	(0,400,0,0,0,0)	(0,2,398,0,0,0,0,0)	(0,0,391,9,0,...)
39000	(0,400)	(5,395,0)	(0,398,2,0,0)	(0,400,0,0,0,0)	(0,1,399,0,0,0,0,0)	(0,0,390,10,0,...)
42000	(0,400)	(1,399,0)	(0,398,2,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,381,19,0,...)
45000	(0,400)	(0,400,0)	(0,398,2,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,387,13,0,...)
48000	(0,400)	(0,400,0)	(0,392,8,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,373,27,0,...)
51000	(0,400)	(1,399,0)	(0,392,8,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,371,29,0,...)
54000	(0,400)	(0,400,0)	(0,393,7,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,373,27,0,...)
57000	(0,400)	(0,400,0)	(0,389,11,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,362,38,0,...)
60000	(0,400)	(0,400,0)	(0,397,3,0,0)	(0,400,0,0,0,0)	(0,0,400,0,0,0,0,0)	(0,0,368,32,0,...)

Table 4.3: The number of colours used by the Heawood colouring algorithm with the colouring process improved using Brooks' recolouring strategy.

for $v = 1$ to m **do**

Set List(v) as a list of all colours not used by the neighbours of v.

end for

while empires are left uncoloured **do**

Set $v =$ uncoloured vertex with shortest list. If two vertices have lists of the same length, choose the one with the most uncoloured neighbours

if List(v) is not empty **then**

Set $c =$ the colour present in lists of the least number of neighbours of v.

Set Colour(v) $= c$.

for all u such that u is a neighbour of v **do**

Set List(u) $=$ List(u)-c.

end for

else

Set Colour(v) $= 2r$.

end if

end while

Return coloured graph.

For this algorithm, each empire has a list of all available colours in the range $1, \ldots, 2r$, initially all colours are available for each empire. At each step, the empire with the shortest list is selected and given the colour in its list that is in the smallest number of lists for neighbouring empires. Unlike the Heawood and Brooks algorithms, it is possible for the list colouring algorithm to use more than $2r$ colours, however tests suggest that this is extremely unlikely.

$n\backslash r$	2	3	4	5	6	10
120	(313,87)	(383,17,0)	(122,275,3,0,0)	(332,59,0,0,0,0)	(197,196,6,0,0,0,0,0)	(199,84,14,0,...)
600	(296,104)	(376,24,0)	(1,398,1,0,0)	(246,154,0,0,0,0)	(3,375,22,0,0,0,0,0)	(49,320,31,0,...)
1200	(273,127)	(366,34,0)	(0,399,1,0,0)	(214,186,0,0,0,0)	(1,379,20,0,0,0,0,0)	(4,317,79,0,...)
3000	(298,111)	(381,19,0)	(0,398,2,0,0)	(184,216,0,0,0,0)	(0,382,18,0,0,0,0,0)	(0,301,99,0,...)
6000	(283,117)	(388,12,0)	(0,398,2,0,0)	(172,228,0,0,0,0)	(0,380,20,0,0,0,0,0)	(0,316,84,0,...)
9000	(282,118)	(372,28,0)	(0,400,0,0)	(187,213,0,0,0,0)	(0,379,21,0,0,0,0,0)	(0,279,121,0,...)
12000	(291,109)	(378,22,0)	(0,398,2,0,0)	(181,219,0,0,0,0)	(0,380,20,0,0,0,0,0)	(0,277,123,0,...)
15000	(284,116)	(372,28,0)	(0,398,2,0,0)	(185,215,0,0,0,0)	(0,388,12,0,0,0,0,0)	(0,270,130,0,...)
18000	(292,108)	(382,18,0)	(0,400,0,0,0)	(172,228,0,0,0,0)	(0,389,11,0,0,0,0,0)	(0,293,107,0,...)
21000	(277,123)	(382,18,0)	(0,399,1,0,0)	(189,211,0,0,0,0)	(0,388,12,0,0,0,0,0)	(0,302,98,0,...)
24000	(285,115)	(385,15,0)	(0,398,2,0,0)	(165,235,0,0,0,0)	(0,373,27,0,0,0,0,0)	(0,289,111,0,...)
27000	(297,103)	(384,16,0)	(0,399,1,0,0)	(172,228,0,0,0,0)	(0,381,19,0,0,0,0,0	(0,291,109,0,...)
30000	(282,118)	(368,32,0)	(0,399,1,0,0)	(167,233,0,0,0,0)	(0,374,26,0,0,0,0,0)	(0,292,108,0,...)
33000	(303,97)	(377,23,0)	(0,399,1,0,0)	(176,224,0,0,0,0)	(0,368,32,0,0,0,0,0)	(0,290,110,0,...)
36000	(275,125)	(381,19,0)	(0,399,1,0,0)	(164,236,0,0,0,0)	(0,380,20,0,0,0,0,0)	(0,289,111,0,...)
39000	(278,122)	(375,25,0)	(0,399,1,0,0)	(167,233,0,0,0,0)	(0,379,21,0,0,0,0,0)	(0,285,115,0,...)
42000	(285,115)	(375,25,0)	(0,398,2,0,0)	(170,230,0,0,0,0)	(0,385,15,0,0,0,0,0)	(0,278,122,0,...)
45000	(288,112)	(384,16,0)	(0,400,0,0,0)	(168,232,0,0,0,0)	(0,378,22,0,0,0,0,0)	(0,270,130,0,...)
48000	(275,125)	(365,35,0)	(0,399,1,0,0)	(167,233,0,0,0,0)	(0,380,20,0,0,0,0,0)	(0,274,126,0,...)
51000	(281,119)	(377,23,0)	(0,398,2,0,0)	(164,236,0,0,0,0)	(0,385,15,0,0,0,0,0)	(0,283,117,0,...)
54000	(268,132)	(380,20,0)	(0,400,0,0,0)	(170,230,0,0,0,0)	(0,381,19,0,0,0,0,0)	(0,271,129,0,...)
57000	(267,133)	(381,19,0)	(0,399,1,0,0)	(159,241,0,0,0,0)	(0,373,27,0,0,0,0,0)	(0,254,146,0,...)
60000	(278,122)	(380,20,0)	(0,399,1,0,0)	(168,232,0,0,0,0)	(0,384,16,0,0,0,0,0)	(0,281,119,0,...)

Table 4.4: The number of colours used by the list colouring algorithm.

Table 4.4 shows the number of colours used by the list colouring algorithm. This algorithm seems to give an improvement over the previous two algorithms, as even with $r = 2$ it avoids using the maximum number of colours on many graphs. For $r = 10$ the algorithm used only eight or nine colours for most graphs, less than half the upper bound.

Figure 4.3 compares the average number of colours used by the list colouring algorithm (bottom diagram) with those used by the other two algorithms we investigated earlier on.

4.3 Short Cycle Preprocessing

A further improvement on the performances of the colouring algorithms that we have discussed so far can be made using a method suggested by Lingsheng Shi and Nicholas Wormald to give asymptotic bounds on the chromatic numbers of random r-regular graphs [64, 65] for small fixed values of r.

The starting point is the observation that any even length cycle can be coloured using two colours only whereas odd cycles need three colours. Adding to this the fact that, in the reduced graphs of random trees, short cycles are relatively few and far apart (see our analysis in Section 3.4.4 and the discussion in Section 3.5.3), one may reasonably conceive that, by first colouring every cycle of at most a given length A, we may devise a more effective colouring heuristic.

We next describe an algorithm, Cycle($\mathcal{H}, A$), that takes as input an uncoloured graph $\mathcal{H}$ containing no loops or parallel edges and an integer $A \geq 3$. As output, the algorithm returns a partial colouring of $\mathcal{H}$ such that all vertices within cycles of length at most A are coloured. To simplify the process we assume that, if one vertex is in at least two cycles, or two vertices from different cycles are adjacent, then the colouring fails and the algorithm is aborted. In what follows vertices are considered *Bad* if they have already been coloured or are adjacent to a coloured vertex, the algorithm fails if

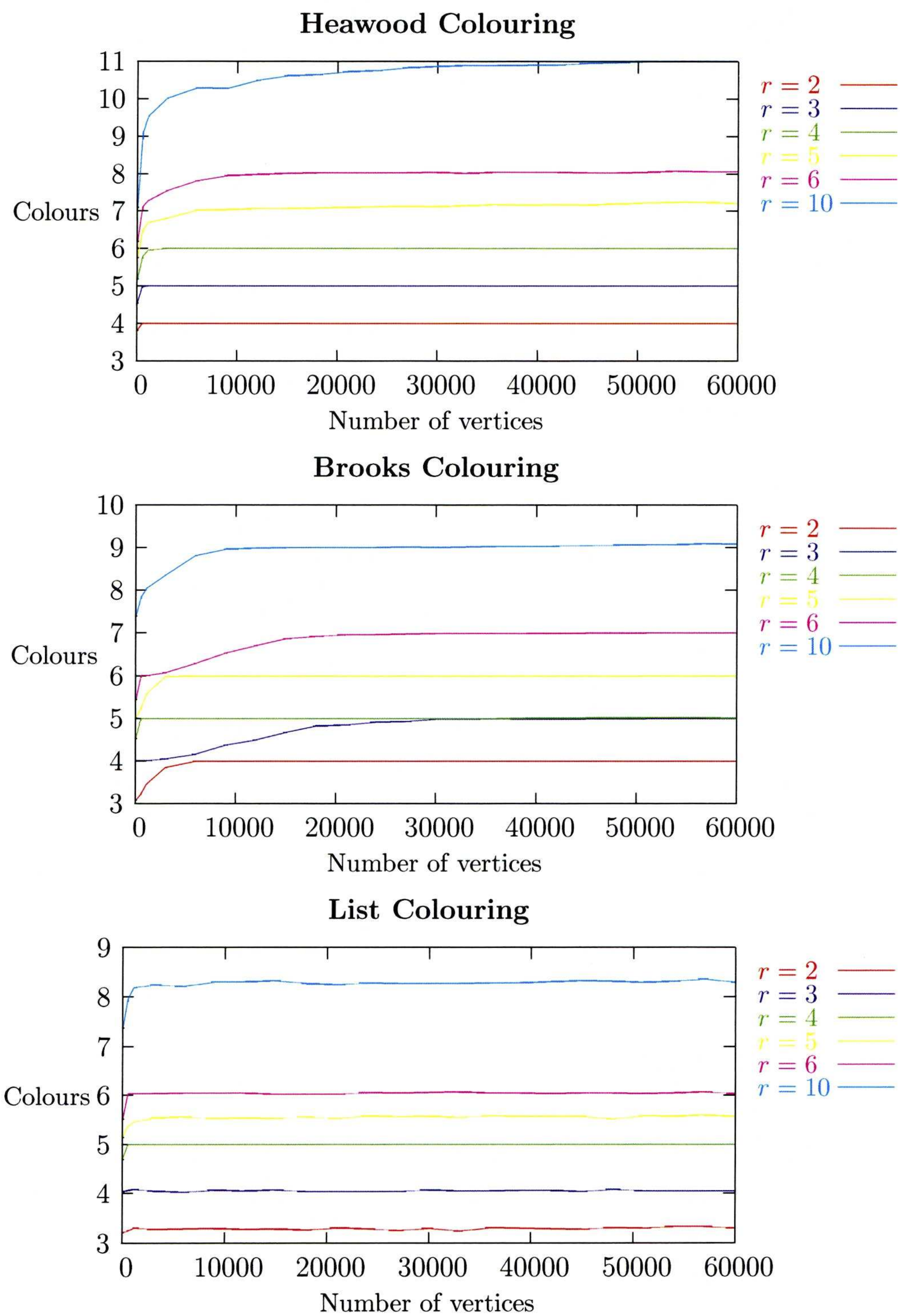

Figure 4.3: The average number of colours used by each of the three algorithms described in this section to colour the r-reduced graphs of trees on n vertices for $2 \leq r \leq 6$ and for $r = 10$.

asked to colour a Bad vertex. We believe that, in analogy to what happens for random regular graphs, because $R_r(\mathcal{T}_n)$ contains relatively few short cycles (see Theorem 3.17 in Chapter 3), for sufficiently large n it is unlikely that two will be intersecting or adjacent and therefore the process simplification described above only marginally affects the performances of the whole heuristic.

The algorithm works by looking at all paths of length $3 \leq l \leq A$ starting at each vertex and seeing if they form a cycle.

Algorithm Cycle($\mathcal{H}$, A)
  for $v = 1$ to n/r **do**
    for $l = 3$ to A **do**
      for all paths $p \in \mathcal{H}$ of length l, starting at v **do**
        if the final vertex of p is v **then**
          if the cycle p has not already been found **then**
            if p contains a Bad vertex **then**
              Colouring fails.
            else
              Colour the vertices of p.
              Set the vertices of p and their neighbours as Bad.
            end if
          end if
        end if
      end for
    end for
  end for
  Return partially coloured graph $\mathcal{H}$.

We can use this short cycle preprocessing algorithm together with list colouring to give an improved algorithm for colouring graphs.

Algorithm ListCycle($\mathcal{H}$)

Set A to some VERY slow growing function of $m = |V(\mathcal{H})|$

if $A < 3$ **then**

$\mathcal{G} = \mathcal{H}$

else

$\mathcal{G} =$ Cycle($\mathcal{H}$,A)

end if

Return List($\mathcal{G}$)

Table 4.5 reports the results obtained by running Cycle($\mathcal{H}$,A) and then the list colouring algorithm described in Section 4.2.3 on the partially coloured graph resulting from Cycle($\mathcal{H}$,A) for $r = 2$ and different values of n and A.

$n \backslash A$	0	3	4	5	6
5000	(296,104)	(278,46)	(14,3)	(0,0)	(0,0)
10000	(293,107)	(302,59)	(59,8)	(0,0)	(0,0)
15000	(284,116)	(299,70)	(97,16)	(0,0)	(0,0)
20000	(291,109)	(328,52)	(125,21)	(0,0)	(0,0)
25000	(284,116)	(328,54)	(154,14)	(0,0)	(0,0)
30000	(282,118)	(321,65)	(188,25)	(1,0)	(0,0)
35000	(296,104)	(316,69)	(187,27)	(1,0)	(0,0)
40000	(291,109)	(334,56)	(225,30)	(1,0)	(0,0)
45000	(292,108)	(316,78)	(224,25)	(0,0)	(0,0)
50000	(271,129)	(323,65)	(240,30)	(4,0)	(0,0)
55000	(281,119)	(320,66)	(229,40)	(2,0)	(0,0)
60000	(276,124)	(326,66)	(244,34)	(7,1)	(0,0)
65000	(275,125)	(329,64)	(258,47)	(7,1)	(0,0)
70000	(291,109)	(332,60)	(252,35)	(5,1)	(0,0)
75000	(284,116)	(327,67)	(275,33)	(17,1)	(0,0)
80000	(290,110)	(329,66)	(263,40)	(17,2)	(0,0)
85000	(297,103)	(321,74)	(278,26)	(16,2)	(0,0)
90000	(284,116)	(327,70)	(278,36)	(24,0)	(0,0)
95000	(277,123)	(329,68)	(282,40)	(27,1)	(0,0)
100000	(279,121)	(342,53)	(290,40)	(20,0)	(0,0)
120000	(278,122)	(329,67)	(232,41)	(40,4)	(0,0)
150000	(284,116)	(307,88)	(310,41)	(59,2)	(0,0)
200000	(277,123)	(330,67)	(318,37)	(103,8)	(0,0)
Prop. 3-coloured	71.23%	83.23%	87.94%	93.85%	—

Table 4.5: The number of trees on n vertices whose 2-reduced graphs were properly three or four coloured by the list colouring algorithm with short cycle preprocessing for cycles of length up to 6. 400 tests performed in each case. Failed tests do not contribute to the tallies and so the numbers in each element do not always add up to the number of tests. The last row gives the proportion of all successful tests leading to graphs that were 3-coloured.

The results suggest that larger values of A may allow more graphs to be coloured with just three colours. Following Shi and Wormald's we conjecture that the analysis of the following strategy may lead to a proof that, in fact, $\chi(R_2(\mathcal{T}_n)) = 3$ a.a.s.

4.4 Conclusions

In this chapter we studied the empire colouring problem on trees. First we proved that, for each positive integer r and n such that $n \geq r$ and n/r is a positive integer, if T_n is a tree on n vertices, the chromatic number of $R_r(T_n)$ is no larger than $2r$ and sometimes $2r$ colours are necessary.

Then we studied empirically a number of colouring heuristics. The results of our analysis can be summarised as follows. First, even a rudimentary minimum degree heuristic is quite good. The number of colours used is never larger than $r+3$. Second, including a re-colouring heuristic similar to the one used in the proof of Brooks theorem seems to further improve the colouring results. The maximum number of used colours for $r = 4$, $r = 5$, and $r = 10$ is always at least one less than in the case of the minimum degree heuristic mentioned before. Overall, the list colouring algorithm described in Section 4.2.3 seems to be the most effective of the three algorithms considered. Finally we argued that some kind of short cycle preprocessing may noticeably improve the performances of the algorithms described, at least for sufficiently large values of n.

We should note that the results obtained using the algorithms in this chapter are promising even just from the combinatorial point of view. For $r > 2$, any of the heuristics considered very rarely required $2r$ colours to complete its job. This leads to the conjecture that $2r$ is a weak upper bound on the chromatic number of the r-reduced graph of a random tree and that a careful combinatorial analysis of the number of s-colourings of $R_r(\mathcal{T}_n)$ may lead to better results than those stated in Theorem 4.1.

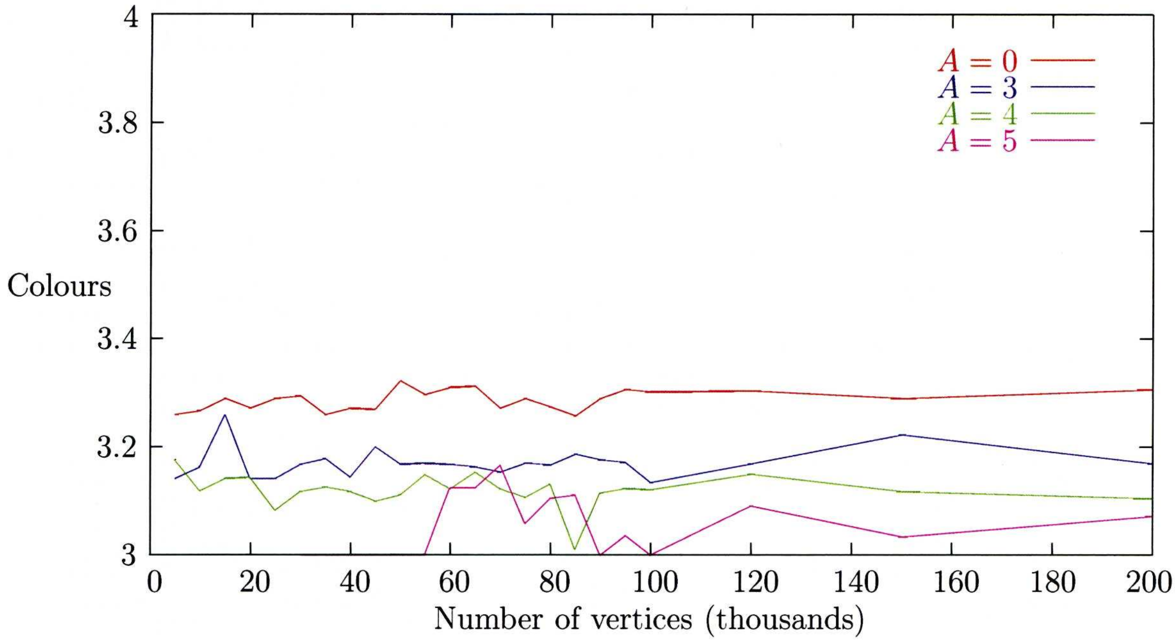

Figure 4.4: The average number of colours used by the list colouring algorithm to colour the 2-reduced graphs of trees on $n = 5000 \ldots 200,000$ vertices with short cycle preprocessing for cycles of length up to 5.

Chapter 5

Bounds on $\chi(R_r(\mathcal{T}_n))$

Let $W_{r,s}(\mathcal{T}_n) = |\mathrm{C}(\mathcal{T}_n, r, s)|$ be the number of proper s-empire colourings of a tree $\mathcal{T}_n$ whose vertex set is partitioned into blocks of size r. Also, define a colouring to be *balanced* if all colour classes are the same size (we will, often tacitly, consider only values of n that make this possible) and let $W'_{r,s}(\mathcal{T}_n)$ be the number of such balanced s-colourings of $\mathcal{T}_n$.

The research question that lead to the results presented in this chapter asked for the asymptotic distribution of $W_{r,s}(\mathcal{T}_n)$ in the hope of pin-pointing, at least for sufficiently large n, the most likely values of $\chi(R_r(\mathcal{T}_n))$, the chromatic number of $R_r(\mathcal{T}_n)$ (or equivalently the minimum positive value of s such that $\mathrm{C}(\mathcal{T}_n, r, s)$ is non-empty). Although we fell short of fulfilling this plan, we managed to characterise all central moments of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$. This in turn leads us to some information on our original questions.

For each fixed integer $r \geq 2$ and $s \geq 1$, let

$$c_{r,s} = s^{\frac{1}{r}-1}(s-1) \tag{5.1}$$

Define s_r as the largest positive integer s such that $c_{r,s} < 1$. The main results in this chapter are as follows:

Theorem 5.1 *For any fixed integer $r \geq 2$, and any $s \in \{1, \ldots, s_r\}$,*

$$W_{r,s}(\mathcal{T}_n) = 0 \qquad a.a.s.$$

Furthermore for large r,

$$s_r = \left\lceil \frac{r}{\log r} \right\rceil \left(1 + O\left(\frac{1}{\log \log r}\right)\right).$$

Table 5.1 gives the values of s_r for the first few values of r.

Theorem 5.1 can be paired up with the following result which, in a sense, gives a measure of its tightness. Informally, for any fixed $r > 1$, and s sufficiently large, if n is large enough, $\mathrm{C}(\mathcal{T}_n, r, s)$ will be non-empty with positive constant probability.

Theorem 5.2 *For any fixed integers $r \geq 2$, and $s \geq 3$ such that*

$$r < \frac{s}{2} \log(s-1),$$

the probability that $W_{r,s}(\mathcal{T}_n) > 0$ is bounded below by a quantity that approaches

$$\frac{\mathrm{e}^{\frac{s(s-2)(r-1)}{(s-1)^2}} \left(r - 2r(s-1)^2 + (s-1)^4\right)^{\frac{(s-1)^2}{2}}}{s^{(s-1)^2}(s-2)^{(s-1)^2}}$$

as n tends to infinity.

r	2	3	4	5	6	7	8	9	10	...	20	...	50
s_r	2	3	3	4	4	4	5	5	6	...	9	...	17

Table 5.1: Lower bounds on the chromatic number of $R_r(\mathcal{T}_n)$ for different values of r.

The proof of Theorem 5.1 (given in Section 5.2.1) is based on noticing that for values of s and r such that $c_{r,s} < 1$, $\mathbf{E}W_{r,s}(\mathcal{T}_n) = o(1)$. To prove Theorem 5.2 we first obtain asymptotic bounds for the first two moments of $W'_{r,s}(\mathcal{T}_n)$ and then use the Cauchy-Schwarz inequality to give a lower bound on the probability that there is at least one proper balanced colouring. As balanced colourings are a special type of colouring, this also acts as a lower bound on the probability that there is at least one proper colouring.

The rest of the Chapter is structured as follows. In Section 5.1 we will define a family of graphs that are related to this colouring problem, and use a method for enumerating spanning trees of such graphs to give exact expressions for all moments of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$ for all values of r and s greater than one. Then, in Section 5.2, we will use these expressions to give bounds on the first moment of $W_{r,s}(\mathcal{T}_n)$ that will be enough to prove Theorem 5.1. In Section 5.3 we study the second moment of $W'_{r,s}(\mathcal{T}_n)$ and give a proof of Theorem 5.2. The Chapter is then concluded by a section speculating on further extensions of the work presented here, comparisons with similar work in the literature, and open problems.

5.1 The Central Moments of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$

The aim of this section is to present exact formulas for the central moments of the random variables $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$.

As we saw in Chapter 2 (see Section 2.3), if we can write a random variable X as a sum of random indicators, then its k^{th} moment can also be written as a sum of simpler terms (see expression (2.28)). Clearly both $W_{r,s}(\mathcal{T}_n)$ and

$W'_{r,s}(\mathcal{T}_n)$ satisfy this condition. For instance

$$W_{r,s}(\mathcal{T}_n) = \sum_{\alpha} I_\alpha$$

where α is a particular assignment of colours from $\{1, \ldots, s\}$ to a set of n vertices and $I_\alpha = 1$ if α is a proper s-(empire) colouring of $\mathcal{T}_n$, zero otherwise. Thus we may compute, say, $\mathbf{E}W_{r,s}(\mathcal{T}_n)^k$ by listing all k-tuples of s-colourings and counting, for each tuple, how many trees are properly coloured by each of the colourings in the tuple. Crucial to our program is a method for enumerating trees on n vertices that are properly coloured by a given k-tuple of s-colourings. After some preparatory work, such a method is described in Section 5.1.3. Finally, in Section 5.1.4 we state and prove our results on $\mathbf{E}W_{r,s}(\mathcal{T}_n)^k$ and $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^k$.

5.1.1 Colourings as Graph Homomorphisms

Colourings of a graph G can be seen as homomorphisms from G to another graph whose vertices correspond to different colour classes [44]. A similar correspondence holds for k-tuples of colourings.

For any integer $s \geq 2$ and $k \geq 1$, let a vertex of graph $B_{s,k}$ be labelled by a sequence $\imath \equiv (i_1, \ldots, i_k)$ where $i_j \in \{1, \ldots, s\}$ for each $j \in \{1, \ldots, k\}$. When lists of such sequences are needed we will assume they are produced in lexicographic order and will denote the elements of such lists by $\imath(1)$, $\imath(2), \ldots$. Thus $\imath(1) = (1, 1, \ldots, 1)$, $\imath(2) = (1, 1, \ldots, 2)$ and so on. If E is an expression involving $\imath(j)$ for some $j \in \{1, \ldots, s^k\}$, then $\sum_\imath E(\imath)$ (or $\prod_\imath E(\imath)$) is a shorthand for $\sum_{j=1}^{s^k} E(\imath(j))$ (or $\prod_{j=1}^{s^k} E(\imath(j))$). Two vertices of $B_{s,k}$, labelled $\imath$ and $\imath$', are adjacent if and only if $i_j \neq i'_j$ for all j's. Thus, $B_{s,k}$ is an $(s-1)^k$-regular graph on s^k vertices (see Figure 5.1 for a small example).

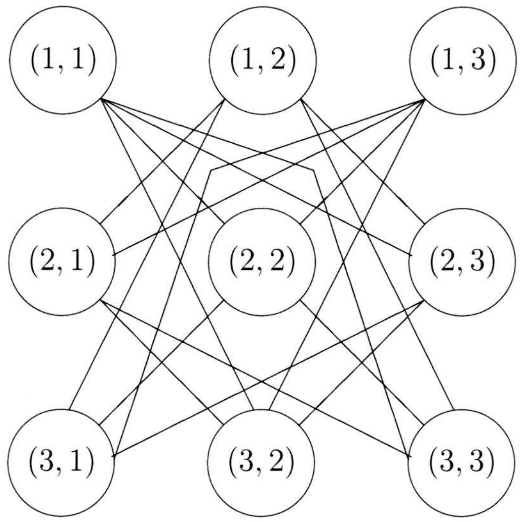

Figure 5.1: The graph $B_{3,2}$ for $s = 3$ and $k = 2$.

Any[1] k-tuple of s-colourings in G defines a homomorphism h from G to $B_{s,k}$: for each $v \in V(G)$, $h(v)$ is the vertex $(i_1, \ldots, i_k) \in V(B_{s,k})$ if v receives colour i_1 by the first given colouring, colour i_2 by the second one and so on. Thus $B_{s,k}$ is referred to as the *constraint graph* on the class of all k-tuples of s-colourings.

The graphs $B_{s,k}$ have a very nice structure and, as we will discover in the next section, also play a role with respect to empire colourings. Before moving to that, we end this section looking at a nice property of constraint graphs that will be used later. In what follows $\kappa(G)$ denotes the number of spanning trees of a graph G.

Lemma 5.3 *For each integer $s \geq 2$ and $k \geq 1$,*

$$s^k \kappa(B_{s,k}) = \prod_{i=1}^{k} ((s-1)^k - (-1)^i (s-1)^{k-i})^{\binom{k}{i}(s-1)^i}.$$

[1]In the current treatment we assume that each of the k colourings uses all s colours available. Strictly speaking an s-colouring does not have to use all s colours available. In fact our analysis will not need this assumption and the way in which we deal which such "efficient" s-colourings is explained in Sections 5.1.2 and 5.1.3.

Proof. By Kirchhoff's matrix-tree theorem (see for example [42, Theorem 1.11, p 29–31]), the number of spanning trees of $B_{s,k}$ is equal to the product of all non-zero eigenvalues of the Laplacian matrix $\mathbf{L}(B_{s,k})$ divided by the number of vertices of $B_{s,k}$. For regular graphs of degree $r > 0$, the spectrum of the Laplacian matrix can be retrieved from that of the corresponding adjacency matrix: if L is an eigenvalue of $\mathbf{A}(B_{s,k})$, then $\lambda = r - L$ is an eigenvalue of $\mathbf{L}(B_{s,k})$. In the rest of the proof we find the spectrum of $\mathbf{A}(B_{s,k})$.

Define the initial matrix $\mathbf{A}(B_{s,0})$ as the 1 by 1 matrix (1). The matrix $\mathbf{A}(B_{s,k+1})$ can be built up recursively from smaller matrices. Any element relating to two vertices with the same first index will be equal to zero and hence there will be s copies of $\mathbf{Zero_{s^k}}$ along the diagonal. Where the first index differs between two matrices the value of the element in $\mathbf{A}(B_{s,k+1})$ depends on whether or not the remaining k indices are the same, this is of course an instance of $\mathbf{A}(B_{s,k})$. Hence, $\mathbf{A}(B_{s,k+1})$ can be obtained from $\mathbf{One_s} - \mathbf{I_s}$ by replacing each zero element by $\mathbf{Zero_{s^k}}$ and each one element by $\mathbf{A}(B_{s,k})$.

$$\mathbf{A}(B_{s,k+1}) = \begin{pmatrix} \mathbf{Zero_{s^k}} & \mathbf{A}(B_{s,k}) & \dots & \mathbf{A}(B_{s,k}) & \mathbf{A}(B_{s,k}) \\ \mathbf{A}(B_{s,k}) & \mathbf{Zero_{s^k}} & & \mathbf{A}(B_{s,k}) & \mathbf{A}(B_{s,k}) \\ \vdots & & \ddots & & \vdots \\ \mathbf{A}(B_{s,k}) & \mathbf{A}(B_{s,k}) & & \mathbf{Zero_{s^k}} & \mathbf{A}(B_{s,k}) \\ \mathbf{A}(B_{s,k}) & \mathbf{A}(B_{s,k}) & \dots & \mathbf{A}(B_{s,k}) & \mathbf{Zero_{s^k}} \end{pmatrix}$$

Given integer i such that $0 \leq i \leq k$, let $l(k,i) = (-1)^i(s-1)^{k-i}$. Note that, for each positive integer k, and $i \in \{0, \dots, k\}$, $l(k,i) = -l(k+1, i+1)$. We will show by induction on k that

$$\mathrm{Spec}\mathbf{A}(B_{s,k}) = \begin{pmatrix} l(k,0) & \dots & l(k,i) & \dots & l(k,k) \\ 1 & \dots & \binom{k}{i}(s-1)^i & \dots & (s-1)^k \end{pmatrix}. \tag{5.2}$$

When $k = 0$, $\mathbf{A}(B_{s,k}) = (1)$. This has one eigenvector (1) with eigenvalue 1, satisfying (5.2). For $k > 0$, we build the eigenvectors of $\mathbf{A}(B_{s,k+1})$ from those of $\mathbf{A}(B_{s,k})$. For each i such that $0 \leq i \leq k$, there are $\binom{k}{i}(s-1)^i$ eigenvectors of $\mathbf{A}(B_{s,k})$ with eigenvalues equal to $l(k,i) = (-1)^i(s-1)^{k-i}$. Let $\mathbf{v}^i$ be one such eigenvector. We build up a new eigenvector $\mathbf{w}^i$ consisting of s copies of $\mathbf{v}^i$. Since $\mathbf{A}(B_{s,k})\mathbf{v}^i = l(k,i)\mathbf{v}^i$, it follows that

$$\mathbf{A}(B_{s,k+1})\mathbf{w}^i = (s-1)l(k,i)\mathbf{w}^i = l(k+1,i)\mathbf{w}^i,$$

as, in each row, $(s-1)$ copies of $\mathbf{v}^i$ are multiplied by $\mathbf{A}(B_{s,k})$ while one is multiplied by zero. This can be done for any of the eigenvectors, giving us $\binom{k}{i}(s-1)^i$ eigenvectors with eigenvalues equal to $l(k+1,i)$ in $\mathbf{A}(B_{s,k+1})$.

Also, for each i such that $0 \leq i \leq k$, given an eigenvector $\mathbf{v}^i$ of $\mathbf{A}(B_{s,k})$ with eigenvalue $l(k,i)$ and integer j with $2 \leq j \leq s$, we can build up another eigenvector $\mathbf{u}^{i,j}$ of $\mathbf{A}(B_{s,k+1})$ defined as follows:

$$\mathbf{u}^{i,j} = \begin{pmatrix} -\mathbf{v}^i \\ \mathbf{z}^2 \\ \mathbf{z}^3 \\ \vdots \\ \mathbf{z}^s \end{pmatrix},$$

where $\mathbf{z}^l = \mathbf{v}^i$ if $l = j$, and a zero vector of length s^k otherwise. When $\mathbf{A}(B_{s,k+1})$ is multiplied by $\mathbf{u}^{i,j}$, the first s^k rows will be equal to $l(k,i)\mathbf{v}^i$ since the $-\mathbf{v}^i$ section was multiplied by zero. The j^{th} set of rows corresponding to $\mathbf{v}^i$ will be equal to $-l(k,i)\mathbf{v}^i$ since the $\mathbf{v}^i$ section was multiplied by zero.

The other rows will be equal to $l(k,i)\mathbf{v}^i - l(k,i)\mathbf{v}^i = 0$. Hence

$$\mathbf{A}(B_{s,k+1})\mathbf{u}^{i,j} = -l(k,i)\mathbf{u}^{i,j} = l(k+1,i+1)\mathbf{u}^{i,j}.$$

For each $\mathbf{v}^i$ there are $(s-1)$ possible places to put the second non-zero block and so we have an additional $\binom{k}{i}(s-1)^{i+1}$ eigenvectors with eigenvalues equal to $l(k+1,i+1)$.

The values $i=0$ and $i=k+1$ in $\mathbf{A}(B_{s,k+1})$ receive eigenvalues from only one place each — $\mathbf{u}^{k,j}$ gives $(s-1)^{k+1}$ eigenvectors with eigenvalues equal to $(-1)^{k+1}$ and $\mathbf{w}^0$ gives one eigenvector with eigenvalue $(s-1)^{k+1}$. For $1 \leq i \leq k$ eigenvalues come from both of the methods described above. Hence, the number of eigenvectors with eigenvalues equal to $l(k+1,i)$ is:

$$\binom{k}{i}(s-1)^i + \binom{k}{i-1}(s-1)^i = \binom{k+1}{i}(s-1)^i.$$

In total, we have defined

$$\sum_{i=0}^{k+1}\binom{k+1}{i}(s-1)^i = s^{k+1}$$

eigenvalues of a square $s^{k+1} \times s^{k+1}$ matrix. The spectrum of $\mathbf{A}(B_{s,k+1})$ is thus fully characterised and the result follows. ■

5.1.2 Operations on Graphs

We define the following operation on finite labelled graphs. Given graphs G and $H_1, \ldots, H_{|V(G)|}$, their *lexicographic product* is the graph, denoted by $G\{H_1,\ldots,H_{|V(G)|}\}$, obtained by replacing the i^{th} vertex of G with H_i for all $i \in \{1,\ldots,|V(G)|\}$. In this process, the first[2] vertex of H_i is re-

[2] We assume that there is a linear order on the vertex labels of the graphs.

labelled $\sum_{j<i} |V(H_j)| + 1$, the second one $\sum_{j<i} |V(H_j)| + 2$, and so on, up until the last vertex which is relabelled $\sum_{j\leq i} |V(H_j)|$. There is an edge in $G\{H_1, \ldots, H_{|V(G)|}\}$ between two vertices $u \in H_i$, $v \in H_j$ $(i \neq j)$ if and only if there is an edge between i and j in G. Two vertices of $G\{H_1, \ldots, H_{|V(G)|}\}$ originating in a given H_i are connected by an edge if and only if they were connected by an edge in H_i.

Note that the definition can be extended to the case where some of the H_i are empty. If H_i is the empty graph we define $G\{H_1, \ldots, H_{i-1}, H_i, H_{i+1}, \ldots, H_{|V(G)|}\}$ as

$$(G - i)\{H_1, \ldots, H_{i-1}, H_{i+1}, \ldots, H_{|V(G)|}\}.$$

In the rest of this thesis we will be interested in one particular instance of the operation defined above. Let r, s, and k be fixed positive integers, with $r \geq 2$ and $s \geq 2$, and let $(m_\imath)_{\imath\in[s]^k}$ be a sequence of s^k non-negative integers (when the fancy indexing of these sequences is not important we will often forget it and denote $(m_\imath)_{\imath\in[s]^k}$ simply using the standard vector notation $\mathbf{m}$). When $N = \sum_\imath m_\imath$ sequences of this kind will be referred to as *s^k-compositions of N*. An s^k-composition will be called *balanced* if for each $j \in \{1, \ldots, k\}$ and $c \in \{1, \ldots, s\}$, the sum of all $m_\imath$ with index $\imath$ such that $i_j = c$, multiplied by s equals N. The graph

$$B_{s,k}\{m_{\imath(1)}K_r, \ldots, m_{\imath(s^k)}K_r\}$$

(here an expression of the form mK_r denotes the graph formed by m disjoint copies of K_r, with $0K_r$ being just another name for the empty graph) has $n = r\sum_\imath m_\imath$ vertices. The graph in the centre of Figure 5.2 is an example of one such construction. The most important property of these graphs is the fact that one can associate a k-tuple of s-colourings of a

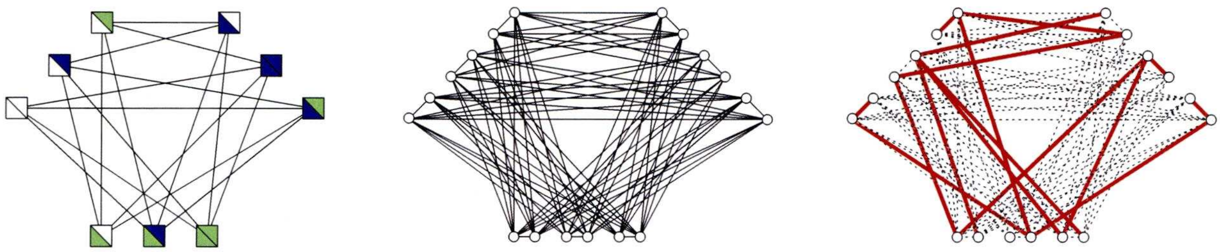

Figure 5.2: The graph $B_{3,2}$ (left, with vertex labels represented by pairs of colours), the graph $B_{3,2}\{1K_2, 1K_2, 1K_2, 1K_2, 1K_2, 1K_2, 1K_2, 1K_2, 1K_2\}$ (centre), and a tree legally coloured by the two colourings (right, in red).

set of n vertices to each graph $B_{s,k}\{m_{\imath(1)}K_r, \ldots, m_{\imath(s^k)}K_r\}$: each vertex originating from a vertex of $m_{\imath(j)}K_r$ is coloured $\imath(j) \equiv (i_1, \ldots, i_k)$. Furthermore this establishes a one-to-one correspondence between the spanning trees of $B_{s,k}\{m_{\imath(1)}K_r, \ldots, m_{\imath(s^k)}K_r\}$ and the trees on n vertices that are properly (empire) coloured by each of the s-colourings in the given k-tuple, and are such that all vertices with labels in $\{1, 2, \ldots, rm_{\imath(1)}\}$ receive colour 1 in each of the k colourings, all vertices with labels in $\{rm_{\imath(1)} + 1, rm_{\imath(1)} + 2, \ldots, r(m_{\imath(1)} + m_{\imath(2)})\}$ receive colour 1 in the first $k - 1$ colourings and 2 in the k^{th} one, and so on. Clearly every spanning tree of $B_{s,k}\{m_{\imath(1)}K_r, \ldots, m_{\imath(s^k)}K_r\}$ is a tree on n vertices that is properly coloured by each of the s-colourings in the k-tuple, in the way mentioned above. Conversely, if there was a tree T on n vertices which was properly coloured by the k given s-colourings in the way mentioned above, but did not span $B_{s,k}\{m_{\imath(1)}K_r, \ldots, m_{\imath(s^k)}K_r\}$ then at least one of its edges, say $e \equiv \{u, v\}$ would join two distinct copies of K_r either in the same $m_{\imath(j)}K_r$ or belonging to $m_{\imath(i'_1)}K_r$ and $m_{\imath(i'_2)}K_r$ such that the tuples $\imath(i'_1)$ and $\imath(i'_2)$ are not pairwise disjoint. But this would imply that for at least one of the k colourings there will be a clash involving the vertices u and v.

5.1.3 Enumeration of Spanning Trees in Certain Classes of Graphs

An enumerative result by M. Rubey [63, Theorem 3.5], based on ideas appearing in earlier work by Knuth [51], Kelmans [50], and Pak and Postnikov [59] gives the number of spanning trees of graphs $G\{H_1, \ldots, H_{|V(G)|}\}$.

To be able to state the result we need to introduce a number of particular expressions. Given graphs G and $H_1, \ldots, H_{|V(G)|}$, let

$$d_G(v) = \sum_{u \in V(G): \{u,v\} \in E(G)} |V(H_u)|$$

be the number of vertices in graphs H_u whose index u, seen as a vertex of G, is adjacent to vertex v. Also, let $f_i(H_v)$ be the number of spanning rooted forests of H_v with i connected components (i.e. the number of spanning forests of H_v consisting of i trees each of which has a specific vertex identified as the root). Let $\overline{K_n}$ denote the edgeless graph on n vertices.

These expressions look quite mysterious, but in all cases relevant to this work they end up having rather simple expressions. So, for instance, if H_u is $\overline{K_{m_{\imath(u)}}}$ for each $u \in \{1, \ldots, s^2\}$, where $r \sum_{\imath} m_{\imath} = n$, then $d_{B_{s,2}}(\imath)$ is equal to

$$\sum_{j_1 : j_1 \neq i_1} \sum_{j_2 : j_2 \neq i_2} m_{(j_1, j_2)},$$

if $\imath = (i_1, i_2)$, or equivalently

$$\frac{n}{r} - \left(\sum_{j=1}^{s} m_{(j, i_2)} + \sum_{j=1}^{s} m_{(i_1, j)} - m_{\imath} \right).$$

Furthermore, if $\sum_{j=1}^{s} m_{(j,i)} = \frac{n}{sr} = \sum_{j=1}^{s} m_{(i,j)}$ (i.e. if the sequence $(m_{\imath})_{\imath \in [s]^2}$

is balanced) then

$$d_{B_{s,2}}(1) = \frac{(s-2)n}{sr} + m_1. \tag{5.3}$$

Similarly, $f_i(H)$ may be difficult to compute for an arbitrary graph H, but fortunately we are only interested in the case $H = K_r$, for some fixed positive integer r. In this case, since the i roots can be chosen in $\binom{r}{i}$ ways it follows that

$$f_i(K_r) = \binom{r}{i} i r^{r-i-1}. \tag{5.4}$$

We are now ready to state the result providing a formula for $\kappa(G\{H_1, \ldots, H_{|V(G)|}\})$.

Theorem 5.4 *Given finite labelled graphs G, and $H_1, \ldots, H_{|V(G)|}$, the number of spanning trees of $G\{H_1, \ldots, H_{|V(G)|}\}$ is equal to*

$$\prod_{v \in V(G), H_v \neq \emptyset} \left(\sum_{i=1}^{|V(H_v)|} f_i(H_v) d_G(v)^{i-1} \right) \sum_T \prod_{v \in V(G), H_v \neq \emptyset} |V(H_v)|^{\deg_T(v)-1}, \tag{5.5}$$

where the second sum is over all spanning trees T of $G[U]$, where $U = \{v \in V(G) : H_v \neq \emptyset\}$.

The careful reader will have noticed that, in fact, Theorem 5.4 is a minor extension of Rubey's result. In our definition of $G\{H_1, \ldots, H_{|V(G)|}\}$ some of the H_i's may be empty. The result, whose proof does not require any additional argument, is necessary to account for s-colourings that in fact only use fewer than s distinct non-empty colour classes.

Theorem 5.4 and the correspondence described in Section 5.1.2 between the spanning trees of $B_{s,k}\{m_{1(1)}K_r, \ldots, m_{1(s^k)}K_r\}$ and a particular set of n-vertex trees properly coloured by a given k-tuple of s-colourings allow us to write down a formula for the number of such trees. To prove the result it is convenient to introduce another instance of lexicographic product. In what follows, let $\mathcal{B}$ denote the graph $B_{s,k}\{\overline{K_{m_{\imath(1)}}}, \ldots, \overline{K_{m_{\imath(s^k)}}}\}$. Note that the equation

$$\kappa(\mathcal{B}) = \prod_{\{\imath: m_\imath \neq 0\}} d_{B_{s,k}}(\imath)^{m_\imath - 1} \sum_T \prod_{\{\imath: m_\imath \neq 0\}} (m_\imath)^{\deg_T(\imath) - 1} \tag{5.6}$$

follows easily from Theorem 5.4 after noticing that for each index $\imath$, such that $m_\imath > 0$, $f_i(\overline{K_{m_\imath}}) = 1$ (resp. 0) when $i = m_\imath$ (resp. $i < m_\imath$), since $\overline{K_{m_\imath}}$ contains no edges. Furthermore, clearly,

$$B_{s,k}\{m_{\imath(1)} K_r, \ldots, m_{\imath(s^k)} K_r\} = \mathcal{B}\{\underbrace{K_r, \ldots, K_r}_{n/r \text{ times}}\}. \tag{5.7}$$

Theorem 5.5 *Let k and n be positive integers. Let r and s be positive integers greater than one, and $(m_\imath)_{\imath \in [s]^k}$ be an s^k-composition of n/r. The number of trees on n vertices that are properly coloured by the k-tuple of s-colourings associated with the graph*

$$B_{s,k}\{m_{\imath(1)} K_r, \ldots, m_{\imath(s^k)} K_r\}$$

in such a way that, for each $j \in \{1, \ldots, k\}$, and $i_j \in \{1, \ldots, s\}$ vertices with labels in the set

$$\left\{ r \sum_{\imath' < \imath} m_\imath + 1, \ldots, r \sum_{\imath' \leq \imath} m_\imath \right\}$$

receive colours $(i_1, \ldots, i_k)$ is

$$\prod_{\{\imath: m_\imath \neq 0\}} (r d_{B_{s,k}}(\imath) + r)^{m_\imath (r-1)} (r d_{B_{s,k}}(\imath))^{m_\imath - 1} \sum_T \prod_{\{\imath: m_\imath \neq 0\}} (r m_\imath)^{\deg_T(\imath) - 1}. \tag{5.8}$$

In this expression, function $d_{B_{s,k}}$ is defined w.r.t. the graph $\mathcal{B}$ and the sum is over all spanning trees T of $B_{s,k}[U]$, where $U = \{\imath : m_\imath \neq 0\}$.

Proof. We work with the more convenient representation of $B_{s,k}\{m_{\imath(1)} K_r, \ldots, m_{\imath(s^k)} K_r\}$

defined in (5.7). Since K_r has r vertices, it follows from Theorem 5.4 that the number of spanning trees of $\mathcal{B}\{K_r, \ldots, K_r\}$ is

$$r^{\frac{n}{r}-2} \prod_{l=1}^{\frac{n}{r}} \left(\sum_{i=1}^{r} f_i(l) d_{\mathcal{B}}(l)^{i-1} \right) \kappa(\mathcal{B}),$$

where the term $f_i(l)$ counts the number of spanning forests consisting of i rooted trees in the l^{th} copy of K_r (and is given by (5.4) as we saw earlier on). Hence the number of spanning trees of $B_{s,k}\{m_{1(1)}K_r, \ldots, m_{1(s^k)}K_r\}$ is

$$r^{\frac{n}{r}-2} \prod_{l=1}^{\frac{n}{r}} \left(\sum_{i=1}^{r} \binom{r}{i} i r^{r-2} \left(\frac{d_{\mathcal{B}}(l)}{r} \right)^{i-1} \right) \kappa(\mathcal{B}).$$

Moving r^{r-2} out of the sum leads to

$$r^{(1-\frac{1}{r})n-2} \prod_{l=1}^{\frac{n}{r}} \left(\sum_{i=1}^{r} \binom{r}{i} i \left(\frac{d_{\mathcal{B}}(l)}{r} \right)^{i-1} \right) \kappa(\mathcal{B}),$$

and finally, using the identity

$$\sum_{i=1}^{r} \binom{r}{i} i \left(\frac{d_{\mathcal{B}}(l)}{r} \right)^{i-1} = r \left(1 + \frac{d_{\mathcal{B}}(l)}{r} \right)^{r-1},$$

and then moving another factor involving r out of the product, we get

$$r^{\frac{n}{r}-2} \prod_{l=1}^{\frac{n}{r}} (r + d_{\mathcal{B}}(l))^{r-1} \kappa(\mathcal{B}). \tag{5.9}$$

Finally, notice that as two vertices in $\mathcal{B}$ are adjacent if and only if they are in subgraphs that are adjacent in $B_{s,k}$, all vertices of $\mathcal{B}$ originating from a given empty graph $\overline{K_{m_1}}$ will have exactly the same degree. Hence $d_{\mathcal{B}}(l)$ will take one value, namely $r d_{B_{s,k}}(\imath(1))$, for every $l \in \{1, \ldots, m_{(1,\ldots,1)}\}$, the value $r d_{B_{s,k}}(\imath(2))$ for $l \in \{m_{(1,\ldots,1)} + 1, \ldots, m_{(1,\ldots,2)}\}$, and so on (see Figure 5.3).

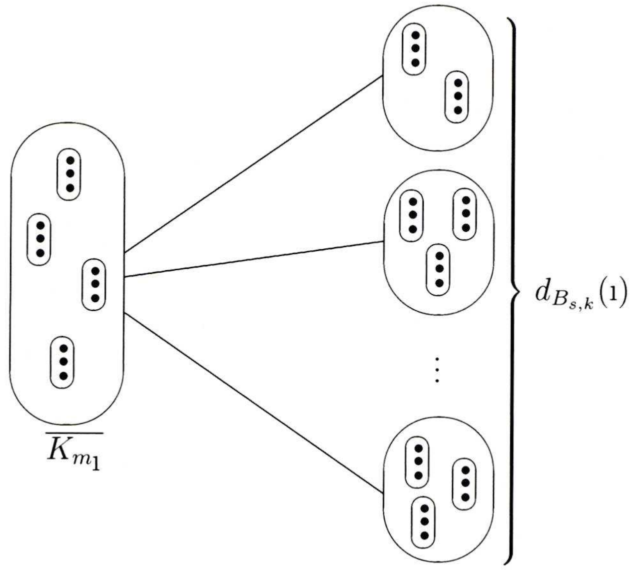

Figure 5.3: The subgraph $\overline{K_{m_\imath}}$ and its neighbours in $\mathcal{B}\{K_r, \ldots, K_r\}$. Every vertex of $\mathcal{B}$ in a given empty subgraph $\overline{K_{m_\imath}}$ is adjacent to the same set of $d_{B_{s,k}}(\imath)$ vertices. As each of these vertices will be replaced by K_r, $d_{\mathcal{B}}(l)$ is equal to $rd_{B_{s,k}}(\imath)$ for any l in $\overline{K_{m_\imath}}$.

Thus we can write:

$$\prod_{l=1}^{\frac{n}{r}} (r + d_{\mathcal{B}}(l))^{r-1} = \prod_{\{\imath : m_\imath \neq 0\}} \left(r + rd_{B_{s,k}}(\imath)\right)^{m_\imath(r-1)}. \tag{5.10}$$

The result follows by putting the values given in equations (5.6) and (5.10) into (5.9).

■

Theorem 5.5 will be needed in the result giving exact formulae for $\mathbf{E}W_{r,s}(\mathcal{T}_n)^k$ and $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^k$. For the asymptotic analysis of Section 5.3 it will also be useful to compare expressions like (5.8) corresponding to different s^k-compositions of n/r. In particular it will be useful to find out under which conditions on the sequence $(m_\imath)_{\imath \in [s]^k}$, the expression (5.8) is asymptotically very close to the one obtained assuming that $m_\imath = \frac{n}{s^k r}$, for all $\imath$. As with the multinomial coefficients in Subsection 2.2.2, it is useful to define for each $\imath$,

$$x_\imath = m_\imath - cn$$

and work with expressions such as

$$\kappa\left(B_{s,k}\{(cn + x_{\imath(1)})K_r, \ldots, (cn + x_{\imath(s^k)})K_r\}\right).$$

For each s^k-tuple $\mathbf{x} = (x_{\imath(1)}, \ldots, x_{\imath(s^k)})$ such that $-cn \leq x_\imath \leq cs^k n - cn$ for all $\imath$ and $\sum_\imath x_\imath = 0$ define

$$g_n(\mathbf{x}) = \frac{\kappa\left(B_{s,k}\{(cn + x_{\imath(1)})K_r, \ldots, (cn + x_{\imath(s^k)})K_r\}\right)}{\kappa\left(B_{s,k}\{cnK_r, \ldots, cnK_r\}\right)} \tag{5.11}$$

(the dependence of g on c is not shown as in all cases c will be fixed). The following result describes the approximation on $g_n(\mathbf{x})$, for $k = 2$, that will be used in Section 5.3.

Lemma 5.6 *Let n be a positive integer. Let r and s be positive integers greater than one. Let c be a fixed positive rational number, and assume that cn is an integer. Finally let $\mathbf{x}$ be a vector formed by s^2 integer numbers such that $-cn \leq x_\imath \leq cs^2 n - cn$ for all $\imath$, satisfying the following conditions:*

1. $\sum_\imath x_\imath = 0$,
2. $\sum_{j=1}^{s} x_{(j,i)} = \sum_{j=1}^{s} x_{(i,j)} = 0$ *for each $i \in \{1, \ldots, s\}$, and*
3. $\max_\imath |x_\imath| = o(n^{\frac{2}{3}})$.

Then

$$g_n(\mathbf{x}) = \exp\left\{\left(\frac{r}{(s-1)^2 cn} - \frac{r}{2(s-1)^4 cn}\right)\sum_\imath x_\imath^2 + O\left(\frac{\max_\imath |x_\imath|^3}{n^2}\right)\right\}$$

as n tends to infinity.

Note that conditions 1. and 2. above imply that, in fact, $g_n(\mathbf{x})$ only depends on $(s-1)^2$ of the s^2 variables. This will be used in Section 5.3, when we will apply Lemma 5.6 to estimate $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$. Here we stick to the current statement to keep the presentation as simple as possible.

Proof of Lemma 5.6. The numerator in the definition of $g_n(\mathbf{x})$ consists of two parts, the product

$$\prod_{1}\left(\sum_{j=1}^{cn+x_1} f_j\left((cn+x_{1(1)})K_r\right) d_{B_{s,2}}(1)^{j-1}\right)$$

and the sum over all spanning trees T of $B_{s,2}$

$$\sum_{T}\prod_{1}(r(cn+x_1))^{\deg_T(1)-1}$$

(the condition $m_1 \neq 0$ can be disregarded as $|x_1|$ is too small for it to happen). In what follows we estimate the ratio of each of these two terms, for arbitrary $\mathbf{x}$ to the corresponding ones for $\mathbf{x}=\mathbf{0}$.

We first look at the sum. Each of its terms can be bounded as follows

$$\left(rcn - r\max_1|x_1|\right)^{s^2-2} \leq \prod_1 (rcn+rx_1)^{\deg_T(1)-1} \leq \left(rcn + r\max_1|x_1|\right)^{s^2-2}$$

while, of course, when $x_1 = 0$,

$$\prod_1 (rcn)^{\deg_T(1)-1} = (rcn)^{s^2-2}.$$

Therefore we have

$$\left(1-\frac{\max_1|x_1|}{cn}\right)^{s^2-2} \leq \frac{\sum_T\prod_1(r(cn+x_1))^{\deg_T(1)-1}}{\sum_T\prod_1(rcn)^{\deg_T(1)-1}} \leq \left(1+\frac{\max_1|x_1|}{cn}\right)^{s^2-2},$$

and, by Lemma 2.1, we can see that

$$\frac{\sum_T \prod_1 (r(cn+x_1))^{\deg_T(1)-1}}{\sum_T \prod_1 (rcn)^{\deg_T(1)-1}} = \exp\left\{ O\left(\frac{\max_1 |x_1|}{n}\right)\right\}.$$

We next look at the product. Theorem 5.5 gives us that

$$\sum_{j=1}^{cn+x_1} f_j\left((cn+x_{1(1)})K_r\right) d_{B_{s,2}}(1)^{j-1} = \left(r d_{B_{s,2}}(1)+r\right)^{(cn+x_1)(r-1)} \left(r d_{B_{s,2}}(1)\right)^{cn+x_1-1}.$$

Using (5.3) we can thus write that the product is equal to

$$\prod_1 \left((s-1)^2 rcn + rx_1 + r\right)^{(cn+x_1)(r-1)} \left((s-1)^2 rcn + rx_1\right)^{cn+x_1-1}.$$

Dividing the terms in the product by their equivalent terms when $\mathbf{x} = \mathbf{0}$ gives

$$\prod_1 \left(1+\frac{x_1}{(s-1)^2 cn}\right)^{cn+x_1-1} \left(1+\frac{x_1}{(s-1)^2 cn+1}\right)^{(cn+x_1)(r-1)}. \tag{5.12}$$

In the remainder of this proof we argue that, under the stated assumptions, expression (5.12) behaves like

$$\prod_1 \left(1+\frac{x_1}{(s-1)^2 cn}\right)^{rcn+rx_1}, \tag{5.13}$$

for n tending to infinity and we prove that the approximation stated on $g_n(\mathbf{x})$ is in fact valid for expression (5.13). To see the latter notice that

$$\prod_1 \left(1+\frac{x_1}{(s-1)^2 cn}\right)^{rcn+rx_1} = \exp\left\{\sum_1 \left[(rcn+rx_1)\log\left(1+\frac{x_1}{(s-1)^2 cn}\right)\right]\right\}.$$

By Lemma 2.2, if we replace each term within the sum with

$$(rcn + rx_1)\left(\frac{x_1}{(s-1)^2cn} - \frac{x_1^2}{2(s-1)^4c^2n^2} + \frac{x_1^3}{3(s-1)^6c^3n^3}\right)$$

and carry out all products we get the following upper bound on (5.13) (remembering that some simplifications occur because of assumption 1. on $\mathbf{x}$).

$$\begin{aligned}
&\exp\left\{\sum_1 \left[(rcn + rx_1)\log\left(1 + \frac{x_1}{(s-1)^2cn}\right)\right]\right\} \\
&\leq \exp\left\{\left(\frac{1}{(s-1)^2} - \frac{1}{2(s-1)^4}\right)\frac{r}{cn}\sum_1 x_1^2 - \left(\frac{1}{2(s-1)^4} - \frac{1}{3(s-1)^6}\right)\frac{r}{(cn)^2}\sum_1 x_1^3\right. \\
&\quad \left.+\frac{r}{3(s-1)^6c^3n^3}\sum_1 x_1^4\right\}.
\end{aligned}$$

The proof of the approximation for (5.13) can be completed using the other half of Lemma 2.2 and assumption 3. on $\mathbf{x}$.

To complete the proof of the lemma we will now argue that expression (5.12) is asymptotically close to (5.13) as n tends to infinity. To see this notice that, if $x = o(n^{2/3})$ then

$$\lim_{n\to\infty}\left(1 + \frac{x}{(s-1)^2cn}\right)^{-1} = 1.$$

Also, for any x and t

$$\begin{aligned}
1 + \frac{x}{t+1} &= \left(1 + \frac{x}{t} + \frac{1}{t}\right)\left(1 - \frac{1}{t+1}\right) \\
&= \left(1 + \frac{x}{t}\right)\left(1 + \frac{1}{t+x}\right)\left(1 - \frac{1}{t+1}\right).
\end{aligned}$$

Using this we see that expression (5.12) is equal to

$$(1 + o(1))\prod_1 \left(1 + \frac{x_1}{(s-1)^2cn}\right)^{rcn+rx_1}$$

multiplied by

$$\prod_{1}\left(1+\frac{1}{(s-1)^2cn+x_1}\right)^{rcn+rx_1}\left(1-\frac{1}{(s-1)^2cn+1}\right)^{rcn+rx_1}. \qquad (5.14)$$

and the claim follows from assumption 3. on $\mathbf{x}$ since expression (5.14) tends to one as n tends to infinity. ■

5.1.4 Main Result

We are now ready to derive an expression for the k^{th} central moment of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$.

For each positive integer k, Theorem 5.5 only counts, via the correspondence described in Section 5.1.2, the number of trees on n vertices that are properly coloured by a given k-tuple of s-colourings, but in such a way that all vertices with labels in $\{1,2,\ldots,rm_{1(1)}\}$ receive colour 1 in each of the k colourings, all vertices with labels in $\{rm_{1(1)}+1, rm_{1(1)}+2,\ldots,r(m_{1(1)}+m_{1(2)})\}$ receive colour 1 in the first $k-1$ colourings and 2 in the k^{th} one, and so on. The proportion of trees that are properly coloured by a k-tuple of s-colourings described by *some* splitting with classes of size $m_{1(1)},\ldots,m_{1(s^k)}$, which we denote by $T_{r,k}(m_{1(1)},\ldots,m_{1(s^k)})$, is just that number divided by n^{n-2} and multiplied by an appropriate multinomial coefficient. After some rearranging this is:

$$\binom{n/r}{m_{1(1)},\ldots,m_{1(s^k)}}\prod_{\{1:m_1\neq 0\}}\left(\frac{rd_{B_{s,k}}(1)+r}{n}\right)^{m_1(r-1)}\left(\frac{rd_{B_{s,k}}(1)}{n}\right)^{m_1-1}\times$$
$$\sum_T\prod_{\{1:m_1\neq 0\}}\left(\frac{rm_1}{n}\right)^{\deg_T(1)-1} \qquad (5.15)$$

where the sum is over all spanning trees T of $B_{s,k}[U]$, where $U=\{1:m_1\neq 0\}$.

The k^{th} central moment of $W_{r,s}(\mathcal{T}_n)$ or $W'_{r,s}(\mathcal{T}_n)$ may then be obtained as a sum of terms of the form (5.15). The difference between the two cases will only be in the number of terms in the sum. This is summarised by the following statement.

Theorem 5.7 *Let k and n be positive integers. Let r and s be positive integers greater than one. Then the k^{th} central moment of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$ are both described by an expression of the form*

$$\sum_{m_{1(1)},\dots,m_{1(s^k)}} T_{r,k}(m_{1(1)},\dots,m_{1(s^k)})$$

In the case of $\mathbf{E}W_{r,s}(\mathcal{T}_n)^k$ (resp. $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^k$) the outer sum is over all sequences $(m_1)_{1\in[s]^k}$ that are s^k-compositions (resp. balanced s^k-compositions) of n/r.

The exact formulae given in Theorem 5.7 represent the starting point of the asymptotic analysis in the following sections. Also, we should point out that the approach used to prove this result significantly generalises the method used in [54] to compute $\mathbf{E}W_{r,s}(\mathcal{T}_n)$. In that paper we relied on the very special structure of $B_{s,1}\{m_1K_r,\dots,m_sK_r\}$ (see Figure 5.4), which makes it easy to find the eigenvalues of its Laplacian matrix. The sought expectation can then be computed resorting, essentially, to the matrix-tree theorem. Unfortunately the method does not scale up, and this leads us to the work presented here.

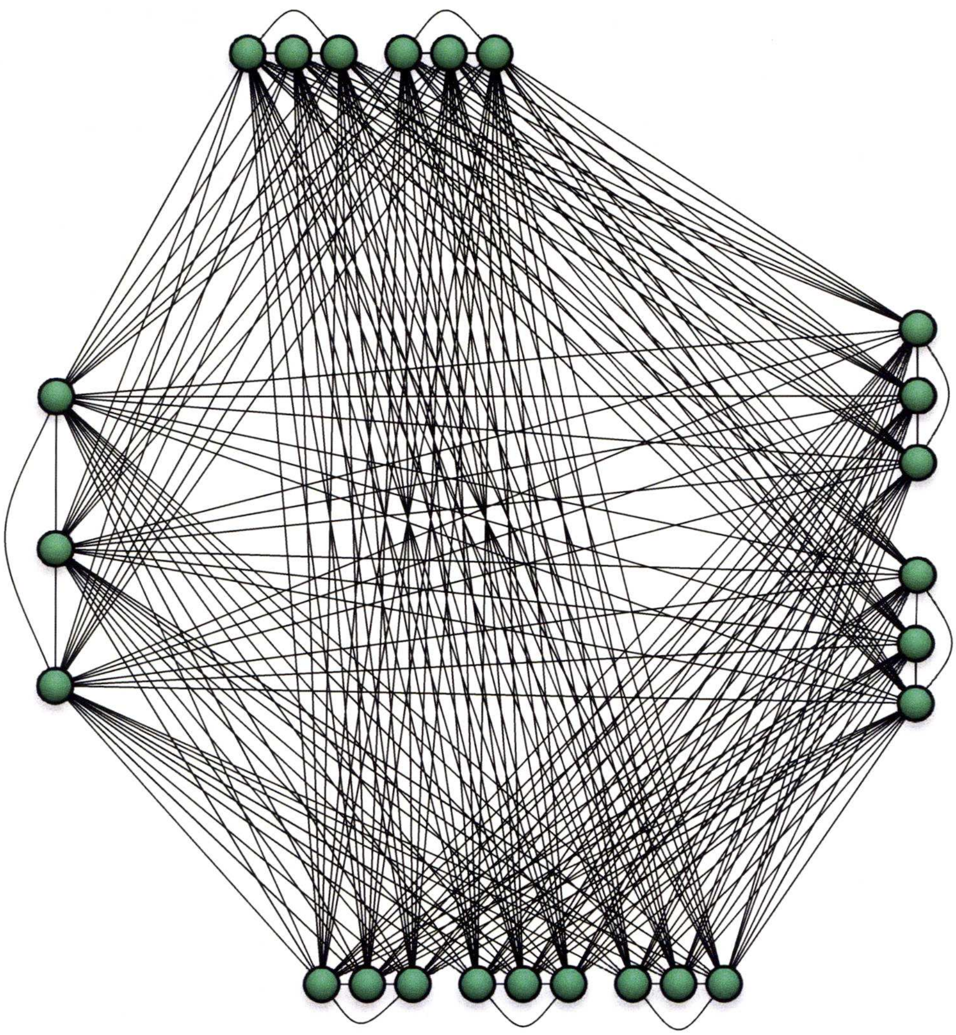

Figure 5.4: The graph $B_{4,1}\{2K_3, 2K_3, 3K_3, 1K_3\}$.

5.2 Approximating the First Moment of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$

In this section we start looking at asymptotic approximations for the moments of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$. We will first concentrate on the expectations. This will serve two purposes. First it will give us what we need to prove Theorem 5.1. Second it will be a warm-up exercise for the more difficult problem of approximating the second moment of $W'_{r,s}(\mathcal{T}_n)$, which will be addressed in Section 5.3.1 and 5.3.2, and will be the key result needed in the proof of Theorem 5.2.

We start with a technical lemma.

Lemma 5.8 *Let s, m and α be positive integers such that m/s is an integer. Let $\mathbf{m}$ be an s-composition of m. Then*

$$\prod_{\{i:m_i\neq 0\}}\left(1+\frac{1}{m}-\frac{m_i}{m}\right)^{\alpha m_i}\left(1-\frac{m_i}{m}\right)^{m_i-1}\leq\left(1+\frac{1}{m}-\frac{1}{s}\right)^{\alpha m}\left(1-\frac{1}{s}\right)^{m-s}.$$

Proof. Firstly, note that when $m_i=0$,

$$\left(1+\frac{1}{m}-\frac{m_i}{m}\right)^{\alpha m_i}\left(1-\frac{m_i}{m}\right)^{m_i-1}=1,$$

and so the product is equal to the product over all i. For each $i\in\{1,\ldots,s\}$, define $x_i=m_i-m/s$, and the *discrepancy* $\operatorname{disc}(\mathbf{m})$ of the sequence $(m_i)_{i=1,\ldots,s}$ to be $\sum_{i=1}^{s}|x_i|$. The result can be proved by induction on $\operatorname{disc}(\mathbf{m})$. If the discrepancy is equal to zero there is nothing to prove. Otherwise we prove that there exists another sequence $\mathbf{m}'=(m_i')_{i=1,\ldots,s}$ with $\operatorname{disc}(\mathbf{m}')<\operatorname{disc}(\mathbf{m})$ and such that

$$\prod_{i=1}^{s}\left(1+\frac{1}{m}-\frac{m_i}{m}\right)^{\alpha m_i}\left(1-\frac{m_i}{m}\right)^{m_i-1}\leq\prod_{i=1}^{s}\left(1+\frac{1}{m}-\frac{m_i'}{m}\right)^{\alpha m_i'}\left(1-\frac{m_i'}{m}\right)^{m_i'-1}.$$

Assume, without loss of generality, that $m_1\geq m_2\geq\ldots\geq m_s$. If the m_i are not all equal to $\frac{m}{s}$ then it must be the case that $m_1-m_s\geq 2$. Define $m_1'=m_1-1$, $m_s'=m_s+1$ and $m_i'=m_i$ for all $i\in\{2,\ldots,s-1\}$. The result will follow from the following inequality:

$$\frac{\left(1+\frac{1}{m}-\frac{m_1}{m}\right)^{\alpha m_1}\left(1-\frac{m_1}{m}\right)^{m_1-1}}{\left(1+\frac{1}{m}-\frac{m'_1}{m}\right)^{\alpha m'_1}\left(1-\frac{m'_1}{m}\right)^{m'_1-1}}\cdot\frac{\left(1+\frac{1}{m}-\frac{m_s}{m}\right)^{\alpha m_s}\left(1-\frac{m_s}{m}\right)^{m_s-1}}{\left(1+\frac{1}{m}-\frac{m'_s}{m}\right)^{\alpha m'_s}\left(1-\frac{m'_s}{m}\right)^{m'_s-1}}\leq 1. \tag{5.16}$$

(notice that as $m'_i = m_i$ for all $i \in \{2, \ldots, s-1\}$, the product over $\mathbf{m}$ is equal to the product over $\mathbf{m}'$ multiplied by (5.16)). The part of (5.16) involving m_1 and m'_1, is equal to

$$\left(\frac{m+1-m_1}{m+1-m'_1}\right)^{\alpha m'_1}\left(\frac{m-m_1}{m-m'_1}\right)^{m'_1-1}\left(1+\frac{1}{m}-\frac{m_1}{m}\right)^{\alpha}\left(1-\frac{m_1}{m}\right)$$

which can be further rewritten as

$$\left(1-\frac{1}{m+1-m'_1}\right)^{\alpha m'_1}\left(1-\frac{1}{m-m'_1}\right)^{m'_1-1}\left(1+\frac{1}{m}-\frac{m_1}{m}\right)^{\alpha}\left(1-\frac{m_1}{m}\right). \tag{5.17}$$

Similarly, the part involving m_s and m'_s is equal to

$$\left(\frac{m+1-m_s}{m+1-m'_s}\right)^{\alpha m_s}\left(\frac{m-m_s}{m-m'_s}\right)^{m_s-1}\left(1+\frac{1}{m}-\frac{m'_s}{m}\right)^{-\alpha}\left(1-\frac{m'_s}{m}\right)^{-1}$$

and, again after trivial manipulations, we get

$$\left(1+\frac{1}{m+1-m'_s}\right)^{\alpha m_s}\left(1+\frac{1}{m-m'_s}\right)^{m_s-1}\left(1+\frac{1}{m}-\frac{m'_s}{m}\right)^{-\alpha}\left(1-\frac{m'_s}{m}\right)^{-1}. \tag{5.18}$$

We will now multiply together the various parts of expressions (5.17) and (5.18) and show that they all come to less than one. We start with the two left-most terms:

$$\left(1-\frac{1}{m+1-m'_1}\right)^{\alpha m'_1}\left(1+\frac{1}{m+1-m'_s}\right)^{\alpha m_s} \leq \left(1-\frac{1}{m+1-m'_1}\right)^{\alpha m'_1}\left(1+\frac{1}{m+1-m'_s}\right)^{\alpha m'_1}$$

$$
\begin{aligned}
&= \left(1 - \frac{m'_1 - m'_s + 1}{(m+1-m'_1)(m+1-m'_s)}\right)^{\alpha m'_1} \\
&< 1.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\left(1 - \frac{1}{m - m'_1}\right)^{m'_1 - 1}\left(1 + \frac{1}{m - m'_s}\right)^{m_s - 1} &\le \left(1 - \frac{1}{m - m'_1}\right)^{m'_1 - 1}\left(1 + \frac{1}{m - m'_s}\right)^{m'_1 - 1} \\
&= \left(1 - \frac{m'_1 - m'_s + 1}{(m - m'_1)(m - m'_s)}\right)^{m'_1 - 1} \\
&< 1.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left(1 + \frac{1}{m} - \frac{m_1}{m}\right)^{\alpha}\left(1 - \frac{m_1}{m}\right)\left(1 + \frac{1}{m} - \frac{m'_s}{m}\right)^{-\alpha}\left(1 - \frac{m'_s}{m}\right)^{-1} &= \\
&= \left(\frac{m + 1 - m_1}{m + 1 - m'_s}\right)^{\alpha}\left(\frac{m - m_1}{m - m'_s}\right),
\end{aligned}
$$

which is smaller than one as $m_1 \ge m'_s$. ∎

We are now ready to prove the main result on $\mathbf{E}W_{r,s}(\mathcal{T}_n)$. Recall that $c_{r,s} = s^{\frac{1}{r}-1}(s-1)$ (as in (5.1).

Theorem 5.9 *Let r and s be positive integers greater than one. Then, for all positive integers n,*

$$
\mathbf{E}W_{r,s}(\mathcal{T}_n) \le \frac{\mathrm{e}^{\frac{s(r-1)}{s-1}} s^s}{(s-1)^s}(c_{r,s})^n. \tag{5.19}
$$

Furthermore, for sufficiently large values of n,

$$
\mathbf{E}W_{r,s}(\mathcal{T}_n) \ge \frac{\mathrm{e}^{\frac{s(r-1)}{s-1} - \frac{rs^2}{n}\left(\frac{r-1}{(s-1)^2} + \frac{1}{12}\right)} s^{\frac{3s}{2}}}{(s-1)^s}\left(\frac{r}{2\pi n}\right)^{\frac{s-1}{2}}(c_{r,s})^n. \tag{5.20}
$$

Proof. Since $B_{s,1} = K_s$, for any given sequence $m_1, \ldots, m_s$ with $n = r \sum_i m_i$, and for any $i \in \{1, \ldots, s\}$,

$$d_{B_{s,1}}(i) = \frac{n}{r} - m_i.$$

Also,

$$\sum_T \prod_{\{i:m_i \neq 0\}} \left(\frac{rm_i}{n}\right)^{\deg_T(i)-1} = \left(\sum_{\{i:m_i\neq 0\}} \frac{rm_i}{n}\right)^{s-2} = 1$$

(the first equality was proved by Rényi [61], the second one is obvious since the m_i add up to n/r). Using all this and Theorem 5.7 we can see that $\mathbf{E}W_{r,s}(\mathcal{T}_n)$ is equal to

$$\sum_{m_1,\ldots,m_s} \binom{n/r}{m_1, \ldots, m_s} \prod_{\{i:m_i\neq 0\}} \left(\left(1 - \frac{rm_i}{n} + \frac{r}{n}\right)^{m_i(r-1)} \left(1 - \frac{rm_i}{n}\right)^{m_i-1}\right), \tag{5.21}$$

the sum being over all s-tuples of non-negative $m_1, \ldots, m_s$ such that $n = r\sum m_i$. In fact, by Lemma 5.8, we can write

$$\begin{aligned}
\mathbf{E}W_{r,s}(\mathcal{T}_n) &\leq \sum_{m_1,\ldots,m_s} \binom{n/r}{m_1, \ldots, m_s} \left(\left(1 - \frac{1}{s} + \frac{r}{n}\right)^{\frac{n}{r}(r-1)} \left(1 - \frac{1}{s}\right)^{\frac{n}{r}-s}\right) \\
&= \sum_{m_1,\ldots,m_s} \binom{n/r}{m_1, \ldots, m_s} \left(\frac{s}{s-1}\right)^s \left(\frac{s-1}{s}\right)^n \left(1 + \frac{rs}{(s-1)n}\right)^{\frac{(r-1)n}{r}} \\
&= s^{\frac{n}{r}} \left(\frac{s}{s-1}\right)^s \left(\frac{s-1}{s}\right)^n \left(1 + \frac{rs}{(s-1)n}\right)^{\frac{(r-1)n}{r}} \qquad (5.22)
\end{aligned}$$

(where the last inequality follows from the generalised version of the binomial theorem, as $\sum_{m_1,\ldots,m_s} \binom{N}{m_1,\ldots,m_s}$ counts the number of ways to partition a set of N elements into s blocks). The bound (5.19) now follows from (5.22) and Lemma 2.1.

A lower bound on (5.21) is given by only considering the term of the sum corresponding to $m_i = \frac{n}{rs}$ for all i. Hence

$$\begin{aligned}
\mathbf{E}W_{r,s}(\mathcal{T}_n) &\geq \binom{n/r}{\frac{n}{rs},\ldots,\frac{n}{rs}} \prod_{\{i:m_i\neq 0\}} \left(\left(1-\frac{1}{s}+\frac{r}{n}\right)^{\frac{n}{rs}(r-1)} \left(1-\frac{1}{s}\right)^{\frac{n}{rs}-1}\right) \\
&= \binom{n/r}{\frac{n}{rs},\ldots,\frac{n}{rs}} \left(\left(1-\frac{1}{s}+\frac{r}{n}\right)^{\frac{n}{r}(r-1)} \left(1-\frac{1}{s}\right)^{\frac{n}{r}-s}\right) \\
&= \binom{n/r}{\frac{n}{rs},\ldots,\frac{n}{rs}} \left(\frac{s}{s-1}\right)^{s} \left(\frac{s-1}{s}\right)^{n} \left(1+\frac{rs}{(s-1)n}\right)^{\frac{(r-1)n}{r}} \qquad (5.23)
\end{aligned}$$

By Stirling's approximations

$$\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}^{\frac{1}{12n}} \qquad (5.24)$$

for any $n \geq 1$ (the two inequalities can be easily derived from the exact expression for $\log(n!)$ given in [36, Chap. 9]). We can use this to prove that, if $n \geq rs$,

$$\binom{\frac{n}{r}}{\frac{n}{rs},\ldots,\frac{n}{rs}} \geq \left(\frac{r}{2\pi n}\right)^{\frac{s-1}{2}} s^{\frac{n}{r}+\frac{s}{2}} \exp\left\{-\frac{rs^2}{12n}\right\}. \qquad (5.25)$$

By Lemma 2.1, for n sufficiently large compared to r and s,

$$\left(1+\frac{rs}{(s-1)n}\right)^{\frac{(r-1)n}{r}} \geq \exp\left\{\frac{s(r-1)}{(s-1)} - \frac{rs^2(r-1)}{(s-1)^2 n}\right\}. \qquad (5.26)$$

The bound (5.20) now follows from (5.23), (5.25), and (5.26). ∎

Before moving to the proof of Theorem 5.1, for completeness, we discuss a much simpler (and tighter) approximation for $\mathbf{E}W'_{r,s}(\mathcal{T}_n)$. The result will be used in the proof of Theorem 5.2. For each integer $r \geq 2$, and $s \geq 2$, let

$$a_n = n^{-\frac{s-1}{2}}(c_{r,s})^n. \qquad (5.27)$$

Lemma 5.10 *Let r and s be positive integers greater than one. Then*

$$\mathbf{E}(W'_{r,s}(\mathcal{T}_n)) \sim \frac{e^{\frac{s(r-1)}{s-1}} s^{\frac{3s}{2}}}{(s-1)^s} \left(\frac{r}{2\pi}\right)^{\frac{s-1}{2}} a_n$$

as n tends to infinity.

Proof. By Theorem 5.7, $\mathbf{E}W'_{r,s}(\mathcal{T}_n) = T_{r,1}\left(\frac{n}{sr}, \ldots, \frac{n}{sr}\right)$. The following equalities simply use the definition of $T_{r,1}\left(\frac{n}{sr}, \ldots, \frac{n}{sr}\right)$:

$$\begin{aligned}
\mathbf{E}W'_{r,s}(\mathcal{T}_n) &= \binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \prod_{\{i:m_i \neq 0\}} \left(\left(1 - \frac{1}{s} - \frac{r}{n}\right)^{\frac{n}{sr}(r-1)} \left(1 - \frac{1}{s}\right)^{\frac{n}{sr}-1}\right) \\
&= \binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \left(1 + \frac{sr}{n(s-1)}\right)^{\frac{n}{r}(r-1)} \left(\frac{s-1}{s}\right)^{n-s}.
\end{aligned}$$

The result now follows from Lemma 2.1, Stirling's approximation (5.24) to handle the factorials involved in $\binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}}$, and simple algebraic manipulations. ■

5.2.1 Lower Bounds on $\chi(R_r(\mathcal{T}_n))$

In this section we give a complete proof of Theorem 5.1. By the Markov inequality,

$$\Pr[W_{r,s}(\mathcal{T}_n) > 0] \leq \mathbf{E}W_{r,s}(\mathcal{T}_n),$$

for any given fixed value of r and s. By inequality (5.19) of Theorem 5.9

$$\mathbf{E}W_{r,s}(\mathcal{T}_n) \leq C(c_{r,s})^n,$$

where C is an expression depending on r and s but independent of n. Since $c_{r,s}$ is a continuous function of r and s and, for fixed $r > 0$ and $s > 1$, it is monotone increasing in s, the first part of the theorem follows easily from the definition of s_r. The rest of the proof gives quantitative estimates on s_r that are valid for sufficiently large (but still fixed) values of r.

Set $s'' = \left(1 + \frac{1}{\log\log r}\right)\frac{r}{\log r}$. Then $c_{r,s''}$ is equal to

$$e^{\frac{1}{r}\log\left(1+\frac{1}{\log\log r}\right)+\frac{\log r}{r}-\frac{\log\log r}{r}}\left(1 - \frac{\log r \log\log r}{r(1+\log\log r)}\right)$$

which, by the lower bound of Lemma 2.1, is at least as large as

$$\left(1 + \frac{\log r}{r} + \frac{1}{r}\log\left(1 + \frac{1}{\log\log r}\right) - \frac{\log\log r}{r}\right)\left(1 - \frac{\log r\log\log r}{r(1+\log\log r)}\right).$$

Since $\log(1+y) > y - \frac{y^2}{2}$ (for any $y < 1$, as stated, say, in [15, page 5]), the expression above is at least as large as

$$\left(1 + \frac{\log r}{r} + \frac{1}{r\log\log r} - \frac{1}{2r(\log\log r)^2} - \frac{\log\log r}{r}\right)\left(1 - \frac{\log r\log\log r}{r(1+\log\log r)}\right).$$

We now multiply things together. The product of the first two monomials in the first term by the second term gives:

$$\begin{aligned} 1 + \frac{\log r}{r} - \frac{\log r}{r}\frac{\log\log r}{1+\log\log r} - \frac{(\log r)^2\log\log r}{r^2(1+\log\log r)} &= \\ = \ 1 + \frac{\log r}{r(1+\log\log r)}\left(1 - \frac{\log r\log\log r}{r}\right) &> 1. \end{aligned}$$

The remaining pieces of the initial product give

$$\begin{aligned} &\left(\frac{1}{r\log\log r} - \frac{1}{2r(\log\log r)^2} - \frac{\log\log r}{r}\right)\left(1 - \frac{\log r\log\log r}{r(1+\log\log r)}\right) = \\ &\quad = \frac{1}{r\log\log r} - \frac{1}{2r(\log\log r)^2} - \frac{\log\log r}{r} - \frac{\log r}{r^2(1+\log\log r)} + \\ &\qquad + \frac{\log r}{2r^2\log\log r(1+\log\log r)} + \frac{\log r(\log\log r)^2}{r^2(1+\log\log r)}. \end{aligned}$$

For large r, all the negative parts on the right-hand side are much smaller than $\frac{\log r}{r(1+\log\log r)}$ and so the lower bound on $c_{r,s''}$ is still larger than one. This argument and the lower bound (5.20) on $\mathbf{E}W_{s.r}(\mathcal{T}_n)$ imply that, for n large enough, the expected number of s-colourings is large provided s is an integer larger than s''.

A similar (but simpler) argument, proves that the expected number of s-colourings is small provided s is small enough. Set $s' = \frac{r}{\log r}$. This time $c_{r,s'}$ is equal to:

$$\mathrm{e}^{\frac{\log r}{r} - \frac{\log\log r}{r}} \left(1 - \frac{\log r}{r}\right) \leq \mathrm{e}^{\frac{\log r}{r}} \left(1 - \frac{\log r}{r}\right).$$

This time we use $e^x \leq 1 + x + x^2 + x^3$ (this is the upper bound of Lemma 2.1, and is valid if $x < 4/7$), to get an upper bound on the expression above. The resulting upper bound is

$$\left(1 + \frac{\log r}{r} + \left(\frac{\log r}{r}\right)^2 + \left(\frac{\log r}{r}\right)^3\right)\left(1 - \frac{\log r}{r}\right) = 1 - \left(\frac{\log r}{r}\right)^4.$$

5.3 Approximating the Second Moment of $W'_{r,s}(\mathcal{T}_n)$

In principle, the calculations used to obtain information about the first moment of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$ could be sharpened and extended to deal with higher moments. However such calculations quickly become very messy as we need to consider all the different ways in which vertices could be coloured in each colouring. In this Section we turn our attention to $W'_{r,s}(\mathcal{T}_n)$, the number of balanced s-colourings of a random tree whose vertex set is subdivided into n/r empires, each of size r. Theorem 5.7 already gives us an exact expression for $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^k$, for each $k \geq 1$. Here we derive approximations for $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ given fixed values of r and s greater than one, as n tends to infinity. Using these and Lemma 5.10 we will finally be able to prove Theorem 5.2.

Recall from equation (5.27) that for each integer $r \geq 2$ and $s \geq 2$, $a_n = n^{-\frac{s-1}{2}}(c_{r,s})^n$, where $c_{r,s} = s^{\frac{1}{r}-1}(s-1)$. The main result of this section is the following:

Theorem 5.11 *Let n be a positive integer. For each integer $r \geq 2$, $s \geq 3$, and such that $r < \frac{s}{2}\log(s-1)$, there exists a function $\Phi_{r,s}(n)$ such that*

$$\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2 \leq \Phi_{r,s}(n)$$

and, furthermore,

$$\Phi_{r,s}(n) \sim \frac{e^{\frac{s^2(r-1)}{(s-1)^2}} s^{s^2+s+1}(s-2)^{(s-1)^2}}{(s-1)^{2s}(r-2r(s-1)^2+(s-1)^4)^{\frac{(s-1)^2}{2}}} \left(\frac{r}{2\pi}\right)^{s-1} \times (a_n)^2$$

as n tends to infinity.

To prove this we will need to argue that the main component of the sum defining $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ consists of all terms close (in a sense that will be made precise later) to the term

$$\overline{\mathbf{m}} = \left(\frac{n}{s^2 r}, \ldots, \frac{n}{s^2 r}\right),$$

where $\overline{\mathbf{m}}_{1(j)} = \frac{n}{s^2 r}$ for all $j \in \{1, \ldots, s^2\}$. More precisely, in Section 5.1.4 we saw that

$$\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2 = \sum_{\mathbf{m}} T_{r,2}(\mathbf{m})$$

where the sum is over all balanced s^2-compositions of n/r. This sum can be split in two parts. A *central* one, over all balanced s^2-compositions whose Euclidean distance from $\overline{\mathbf{m}}$ is at most $\rho(n)$, for some function $\rho(n) = o(n^{2/3})$, and a peripheral one (referred to as the *tail* from now on) over all other compositions. We will argue that the former carries all the useful information about $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$. To this end, first notice that, using definitions (2.2) and (5.11), with $c = (s^2r)^{-1}$, and $t = s^2$, we can write

$$\sum_{\mathbf{m}:||\mathbf{m}-\overline{\mathbf{m}}||_2 \le \rho(n)} T_{r,2}(\mathbf{m}) = T_{r,2}\left(\overline{\mathbf{m}}\right) \times \sum_{\mathbf{x}:||\mathbf{x}||_2 \le \rho(n)} g_n(\mathbf{x}) f_n(\mathbf{x})$$

(where $\mathbf{x} = \mathbf{m} - \overline{\mathbf{m}}$). In the next section, we will:

1. provide an asymptotic expression for $T_{r,2}(\overline{\mathbf{m}})$, and then

2. argue that,

   $$\sum_{\mathbf{x}:||\mathbf{x}||_2 \le \rho(n)} g_n(\mathbf{x}) f_n(\mathbf{x})$$

   is bounded above by a quantity that is very close to

   $$n^{\frac{(s-1)^2}{2}} \int_{\mathbb{R}^{(s-1)^2}} \mathrm{e}^{-\frac{1}{2}\mathbf{y}\mathbf{A}_{\mathbf{r},\mathbf{s}}\mathbf{y}^T} d\mathbf{y} \tag{5.28}$$

   as n tends to infinity (here $\mathbf{A}_{\mathbf{r},\mathbf{s}}$ is an $(s-1)^2 \times (s-1)^2$ non-singular positive definite real symmetric matrix).

Because, modulo some trivial re-scaling, expression (5.28) essentially involves the well-known Gaussian integral, the study of the spectrum of $\mathbf{A}_{\mathbf{r},\mathbf{s}}$ enables us to define an approximation for the central part of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ sufficient to prove Theorem 5.11. The proof of the Theorem is completed by a careful study of the tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ which is carried out in Section 5.3.2.

5.3.1 The Central Part of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$

We start by proving a result about $T_{r,2}(\overline{\mathbf{m}})$.

Lemma 5.12 *Let n be a positive integer and let r and s be positive integers greater than one. Then*

$$T_{r,2}\left(\overline{\mathbf{m}}\right) \sim \frac{\mathrm{e}^{\frac{s^2(r-1)}{(s-1)^2}} s^{s^2+4}}{(s-1)^{2s^2}} \kappa(B_{s,2}) \left(\frac{r}{2\pi n}\right)^{\frac{s^2-1}{2}} (c_{r,s})^{2n}$$

as n tends to infinity.

Proof. From (5.15), we can see that $T_{r,2}\left(\overline{\mathbf{m}}\right)$ is equal to

$$\binom{\frac{n}{r}}{\frac{n}{s^2r}, \ldots, \frac{n}{s^2r}} \prod_{\imath} \left(\frac{rd_{B_{s,2}}(\imath)+r}{n}\right)^{\frac{n}{s^2r}(r-1)} \left(\frac{rd_{B_{s,2}}(\imath)}{n}\right)^{\frac{n}{s^2r}-1} \sum_T \prod_{\imath} \left(\frac{1}{s^2}\right)^{\deg_T(\imath)-1} \tag{5.29}$$

where the sum is over all spanning trees of $B_{s,2}$. Note that, since any spanning tree of $B_{s,2}$ has s^2-1 edges,

$$\prod_{\imath} \left(\frac{1}{s^2}\right)^{\deg_T(\imath)-1} = \left(\frac{1}{s^2}\right)^{2(s^2-1)-s^2}.$$

Thus

$$\sum_T \prod_{\imath} \left(\frac{1}{s^2}\right)^{\deg_T(\imath)-1} = \left(\frac{1}{s^2}\right)^{2(s^2-1)-s^2} \kappa(B_{s,2}). \tag{5.30}$$

Furthermore, since all blocks have the same size, $d_{B_{s,2}}(\imath) = \left(\frac{(s-1)^2 n}{s^2 r}\right)$ for all $\imath$, and hence

$$\prod_{\imath} \left(\frac{rd_{B_{s,2}}(\imath)+r}{n}\right)^{m_{\imath}(r-1)} \left(\frac{rd_{B_{s,2}}(\imath)}{n}\right)^{m_{\imath}-1} =$$

$$= r^{n-s^2}\prod_{\imath}\left(\frac{(s-1)^2}{s^2r}+\frac{1}{n}\right)^{\frac{(r-1)n}{s^2r}}\left(\frac{(s-1)^2}{s^2r}\right)^{\frac{n}{s^2r}-1}$$

$$= r^{n-s^2}\left(\frac{(s-1)^2}{s^2r}\right)^{n-s^2}\left(1+\frac{s^2r}{(s-1)^2n}\right)^{(r-1)\frac{n}{r}}$$

$$= \left(\frac{s}{s-1}\right)^{2s^2}\left(1+\frac{s^2r}{(s-1)^2n}\right)^{(r-1)\frac{n}{r}}\left(\frac{c_{r,s}}{s^{\frac{1}{r}}}\right)^{2n}. \tag{5.31}$$

Thus

$$T_{r,2}(\overline{\mathbf{m}}) = \binom{\frac{n}{r}}{\frac{n}{s^2r},\ldots,\frac{n}{s^2r}}\frac{s^4}{(s-1)^{2s^2}}\,\kappa(B_{s,2})\left(1+\frac{s^2r}{(s-1)^2n}\right)^{(r-1)\frac{n}{r}}\left(\frac{c_{r,s}}{s^{\frac{1}{r}}}\right)^{2n}.$$

The proof is completed using Lemma 2.1 to approximate

$$\left(1+\frac{s^2r}{(s-1)^2n}\right)^{(r-1)\frac{n}{r}}$$

in terms on exponential factions, and the relationship

$$\binom{n/r}{\frac{n}{s^2r},\ldots,\frac{n}{s^2r}} \sim s^{\frac{2n}{r}+s^2}\left(\frac{r}{2\pi n}\right)^{\frac{s^2-1}{2}}$$

(which can be easily derived from Stirling's approximations (5.24)) valid for fixed values of r and s. ■

Next we look at

$$\sum_{\mathbf{x}:||\mathbf{x}||_2\le\rho(n)} g_n(\mathbf{x})f_n(\mathbf{x}).$$

In what follows let $s(\imath)$ be the number of elements equal to s in $\imath$. Because the sum is defined in terms of balanced compositions only, we can bound it above by a sum over all possible values of $x_\imath$ (subject to the constraint $||\mathbf{x}||_2 \le \rho(n)$) summed only for the $(s-1)^2$ indices $\imath$ having $s(\imath)=0$. In fact note that

$$x_{(s,s)} = - \sum_{1 \neq 1(s^2)} x_1 = - \sum_{1:s(1)=0} x_1 - \sum_{1:s(1)=1} x_1 \tag{5.32}$$

and for any $1 \leq l < s$,

$$x_{(l,s)} = - \sum_{i=1}^{s-1} x_{(l,i)} \qquad x_{(s,l)} = - \sum_{i=1}^{s-1} x_{(i,l)}. \tag{5.33}$$

We will now argue that this expression is bounded above by expression (5.28). To this end it is enough to show that, under the stated assumptions, $g_n(\mathbf{x})$ and $f_n(\mathbf{x})$ can be bounded above by some exponential functions only involving the $(s-1)^2$ variables mentioned above. It will be a by-product of our analysis that such functions have the form

$$e^{-\frac{1}{2n}\mathbf{z}\mathbf{M}_{\mathbf{r},\mathbf{s}}\mathbf{z}^T} \tag{5.34}$$

(where $\mathbf{z} \in \mathbb{Z}^{(s-1)^2}$ and $\mathbf{M}_{\mathbf{r},\mathbf{s}}$ is an $(s-1)^2 \times (s-1)^2$ non-singular positive definite real symmetric matrix) and thus only depend on the square of the Euclidean norm of $\mathbf{z}$. This, in turn, implies that

$$\sum_{\mathbf{x}:||\mathbf{x}||_2 \leq \rho(n)} g_n(\mathbf{x}) f_n(\mathbf{x}) \leq \sum_{\mathbf{z} \in \mathbb{Z}^{(s-1)^2}} e^{-\frac{1}{2n}\mathbf{z}\mathbf{A}_{\mathbf{r},\mathbf{s}}\mathbf{z}^T}$$

and now it is not difficult to realise that, after the substitution $\mathbf{y} = \frac{\mathbf{z}}{\sqrt{n}}$ the sum on the right-hand side is $n^{\frac{(s-1)^2}{2}}$ times a Riemann sum for the function $e^{-\frac{1}{2}\mathbf{y}\mathbf{A}_{\mathbf{r},\mathbf{s}}\mathbf{y}^T}$ and, therefore, for n large, well approximated by (5.28). We complete this part of the argument looking at the approximations on $g_n(\mathbf{x})$ and $f_n(\mathbf{x})$. In what follows the expression $\delta_{a,b}$ stands for one (resp. zero) if expressions a and b are equal (resp. different). We remind the reader that, by Lemma 2.5 and 5.6, under the stated assumptions on $\mathbf{x}$, we have that

$$g_n(\mathbf{x}) = (1+o(1))\exp\left\{\left(1-\frac{1}{2(s-1)^2}\right)\frac{s^2r^2}{(s-1)^2n}\sum_{\mathbb{1}} x_{\mathbb{1}}^2\right\}, \quad (5.35)$$

$$f_n(\mathbf{x}) = (1+o(1))\exp\left\{-\frac{s^2r}{n}\sum_{\mathbb{1}:\mathbb{1}\neq \mathbb{1}(s^2)} x_{\mathbb{1}}\left(\sum_{\mathbb{1}':\mathbb{1}'\leq \mathbb{1}} x_{\mathbb{1}'}\right)\right\}. \quad (5.36)$$

The fact that, when $\mathbf{x}$ are such that $||\mathbf{x}||_2 \leq \rho(n)$, $g_n(\mathbf{x})$ and $f_n(\mathbf{x})$ are approximated by expressions of the form (5.34) is a direct consequence of the assumptions of Lemma 2.5 and 5.6, stating that the approximations (5.35) and (5.36) are valid provided $\max_{\mathbb{1}} x_{\mathbb{1}} = ||\mathbf{x}||_\infty = o(n^{2/3})$ and $o(n)$ respectively, the fact that $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$ and the following property of the sequences $(\mathbf{x}_{\mathbb{1}})_{\mathbb{1}\in[s]^2}$.

Lemma 5.13 *Let s be a fixed integer greater than one, and let $(\mathbf{x}_{\mathbb{1}})_{\mathbb{1}\in[s]^2}$ be a sequences of integer numbers satisfying:*

1. $\sum_{\mathbb{1}} x_{\mathbb{1}} = 0$, *and*

2. $\sum_{j=1}^{s} x_{(j,i)} = \sum_{j=1}^{s} x_{(i,j)} = 0$ *for each* $i \in \{1,\dots,s\}$.

Then

$$\sum_{\mathbb{1}} x_{\mathbb{1}}^2 = 2\sum_{\mathbb{1}:\mathbb{1}\neq \mathbb{1}(s^2)} x_{\mathbb{1}}\left(\sum_{\mathbb{1}':\mathbb{1}'\leq \mathbb{1}} x_{\mathbb{1}'}\right) \quad (5.37)$$

and

$$\sum_{\mathbb{1}:\mathbb{1}\neq \mathbb{1}(s^2)} x_{\mathbb{1}}\left(\sum_{\mathbb{1}':\mathbb{1}'\leq \mathbb{1}} x_{\mathbb{1}'}\right) = \frac{1}{2}\sum_{\mathbb{1}:s(\mathbb{1})=0}\sum_{\mathbb{1}':s(\mathbb{1}')=0}(1+\delta_{i_1,i_1'}+\delta_{i_2,i_2'}+\delta_{\mathbb{1},\mathbb{1}'})x_{\mathbb{1}}x_{\mathbb{1}'}. \quad (5.38)$$

Proof. To see the first identity, note that we can write

$$\sum_{\mathbb{1}} x_{\mathbb{1}}^2 = \sum_{\mathbb{1}:s(\mathbb{1})=0} x_{\mathbb{1}}^2 + \sum_{\mathbb{1}:s(\mathbb{1})=1} x_{\mathbb{1}}^2 + x_{(s,s)}^2.$$

Using (5.32), we then have

$$\sum_{1} x_1^2 = \sum_{1:s(1)=0} x_1^2 + \sum_{1:s(1)=1} x_1^2 + \left(\sum_{1:s(1)=0} x_1 + \sum_{1:s(1)=1} x_1 \right)^2$$

and the result follows from the multinomial theorem.

The second one follows, similarly, using identity (5.33) to get rid of all x_1 having $s(1) = 1$. More specifically we can write:

$$\begin{aligned}
&\sum_{1:1\neq 1(s^2)} x_1 \left(\sum_{1':1'\leq 1} x_{1'} \right) = \\
&= \frac{1}{2} \left[\sum_{1:1\neq 1(s^2)} x_1^2 + \left(\sum_{1:1\neq 1(s^2)} x_1 \right)^2 \right] \\
&= \frac{1}{2} \left[\sum_{1:s(1)=0} x_1^2 + \sum_{1:s(1)=1} x_1^2 + \left(\sum_{1:s(1)=0} x_1 + \sum_{1:s(1)=1} x_1 \right)^2 \right] \\
&= \frac{1}{2} \left[\left(\sum_{1:s(1)=0} x_1 \right)^2 + \sum_{1:s(1)=0} x_1^2 + \sum_{1:s(1)=1} x_1^2 + \left(\sum_{1:s(1)=1} x_1 \right)^2 + 2 \sum_{1:s(1)=0} x_1 \sum_{1:s(1)=1} x_1 \right].
\end{aligned}$$

To complete the proof of the lemma we claim that

$$\left(\sum_{1:s(1)=0} x_1 \right)^2 + \sum_{1:s(1)=0} x_1^2 = \sum_{1:s(1)=0} \sum_{1':s(1')=0} (1 + \delta_{1,1'}) x_1 x_{1'}, \tag{5.39}$$

$$\sum_{1:s(1)=1} x_1^2 = \sum_{1:s(1)=0} \sum_{1':s(1')=0} (\delta_{i_1,i_1'} + \delta_{i_2,i_2'}) x_1 x_{1'}, \tag{5.40}$$

and finally that

$$\left(\sum_{1:s(1)=1} x_1 \right)^2 + 2 \sum_{1:s(1)=0} x_1 \sum_{1:s(1)=1} x_1 = 0. \tag{5.41}$$

To believe (5.39) simply note that monomials of the form $x_\imath x_{\imath'}$ occur twice in the expression on the left-hand side if and only if $\imath = \imath'$.

Identity (5.40) can be understood similarly after noticing that, using (5.33) repeatedly, we have

$$\sum_{\imath:s(\imath)=1} x_\imath^2 = \sum_{i=1}^{s-1}\left(\sum_{j=1}^{s-1} x_{(i,j)}\right)^2 + \sum_{j=1}^{s-1}\left(\sum_{i=1}^{s-1} x_{(i,j)}\right)^2 .$$

For each given $i \in \{1, \ldots, s-1\}$

$$\left(\sum_{j=1}^{s-1} x_{(i,j)}\right)^2 = \sum_{j_1=1}^{s-1}\sum_{j_2=1}^{s-1} x_{(i,j_1)} x_{(i,j_2)},$$

and similarly, for given $j \in \{1, \ldots, s-1\}$

$$\left(\sum_{i=1}^{s-1} x_{(i,j)}\right)^2 = \sum_{i_1=1}^{s-1}\sum_{i_2=1}^{s-1} x_{(i_1,j)} x_{(i_2,j)}.$$

This implies that the only monomials of the form $x_\imath x_{\imath'}$ which contribute to the sum on the left-hand side of (5.40) are those sharing at least one element of their index. Furthermore monomials of the form $x_\imath x_\imath$ contribute twice.

Finally, we look at identity (5.41). We will argue that, after all terms $x_\imath$ with $s(\imath) = 1$ have been replaced by terms whose index does not contain s using (5.33), each monomial of the form $x_\imath^2$ occurs four times and each monomial of the form $x_\imath x_{\imath'}$ with $\imath \neq \imath'$ occurs eight times in

$$\left(\sum_{\imath:s(\imath)=1} x_\imath\right)^2 \tag{5.42}$$

and in

$$2\sum_{\imath:s(\imath)=0} x_\imath \sum_{\imath:s(\imath)=1} x_\imath \tag{5.43}$$

but with opposite signs and therefore the whole thing cancels out. The former expression is equivalent to

$$\sum_{1:s(1)=1} \sum_{1':s(1')=1} x_1 x_{1'}.$$

Table 5.2 gives the contribution to different types of monomials $x_1 x_{1'}$ coming from different types of monomials in the expression (5.42). Note that the

	$x_{(i,j)}x_{(i',j')}$	$x_{(i,j)}x_{(i,j')}$	$x^2_{(i,j)}$
$x_{(i,s)}x_{(i',s)}$	2		
$x_{(s,j)}x_{(s,j')}$	2	2	
$x_{(s,j)}x_{(i',s)}$	2		
$x_{(s,j')}x_{(i,s)}$	2	2	
$x_{(s,j)}x_{(i,s)}$		2	2
$x_{(i,s)}x_{(i,s)}$		2	1
$x_{(s,j)}x_{(s,j)}$			1

Table 5.2: Coefficients of $x_{(i,j)}x_{(i',j')}$ in the expansion of (5.42).

twos in the first column, in all but the last entry in column two, and the only two in column three, are due to the fact that the expression above contains both $x_1 x_{1'}$ and $x_{1'} x_1$. On the other hand, the two on the line labelled by $x_{(i,s)}x_{(i,s)}$ is due to the fact that $x_{(i,j)}x_{(i,j')}$ has coefficient two in the product

$$\sum_{j=1}^{s-1} x_{(i,j)} \times \sum_{j=1}^{s-1} x_{(i,j)}.$$

As to (5.43), using repeatedly (5.33) we have

$$\begin{aligned}
\sum_{1:s(1)=1} x_1 &= \sum_{i=1}^{s-1} x_{(i,s)} + \sum_{j=1}^{s-1} x_{(s,j)} \\
&= -\sum_{i=1}^{s-1}\sum_{j=1}^{s-1} x_{(i,j)} - \sum_{j=1}^{s-1}\sum_{i=1}^{s-1} x_{(i,j)} \\
&= -2 \sum_{1:s(1)=0} x_1
\end{aligned}$$

Hence

$$
\begin{aligned}
2 \sum_{1:s(1)=0} x_1 \sum_{1:s(1)=1} x_1 &= -4 \left(\sum_{1:s(1)=0} x_1 \right)^2 \\
&= -4 \sum_{1:s(1)=0} x_1^2 - 8 \underbrace{\sum_{1:s(1)=0} \sum_{1':s(1')=0}}_{1 \neq 1'} x_1 x_{1'}
\end{aligned}
$$

which completes the proof of identity (5.41). ■

We complete this section looking at the integral

$$\int_{\mathbb{R}^{(s-1)^2}} \mathrm{e}^{-\frac{1}{2}\mathbf{y}\mathbf{A}_{\mathbf{r},\mathbf{s}}\mathbf{y}^T} d\mathbf{y}.$$

By the change of variable Theorem (see Lemma 2.6 in Section 2.2.3)

$$\int_{\mathbb{R}^{(s-1)^2}} \mathrm{e}^{-\frac{1}{2}\mathbf{y}\mathbf{A}_{\mathbf{r},\mathbf{s}}\mathbf{y}^T} d\mathbf{y} = (2\pi)^{\frac{(s-1)^2}{2}} \prod_{i=1}^{(s-1)^2} \frac{1}{\sqrt{\lambda_i}}$$

where λ_i are the eigenvalues of $\mathbf{A}_{\mathbf{r},\mathbf{s}}$. Lemma 2.6 also gives us a way to compute the eigenvalues of $\mathbf{A}_{\mathbf{r},\mathbf{s}}$ and thus get an explicit expression for the integral above. The argument is spelled out in the rest of the section.

For the remainder of this section we will refer to the elements of matrices (resp. vectors) using notation of the form $(\mathbf{A})_{1,1'}$ (resp. $\mathbf{v}_1$), where $1 = \{i_1, i_2\}$ and $1' = \{i'_1, i'_2\}$ are ordered 2-tuples with elements in the range $\{1, \ldots s-1\}$ (as defined in subsection 5.1.1). By the lexicographic ordering of these tuples, this is equivalent to writing $(\mathbf{A})_{(s-1)(i_1-1)+i_2,(s-1)(i'_1-1)+i'_2}$ in standard matrix notation. The reasoning behind this notation is that the values of i_1, i_2, i'_1 and i'_2 will often be important in determining the values of elements of matrices and vectors.

For positive integers s, and r greater than one define

$$X_{r,s} = s^2 r \left(\frac{r - 2r(s-1)^2 + (s-1)^4}{(s-1)^4} \right).$$

Let $\mathbf{A'_s}$ be the $(s-1)^2 \times (s-1)^2$ real symmetric matrix defined through the equation

$$(\mathbf{A'_s})_{1,1'} = (1 + \delta_{i_1,i'_1} + \delta_{i_2,i'_2} + \delta_{1,1'}).$$

It follows from the approximations (5.35) and (5.36) and from Lemma 5.13 that

$$\mathbf{A_{r,s}} = \mathbf{A'_s} X_{r,s}. \tag{5.44}$$

So, for instance, $\mathbf{A_{2,2}}$ is just the single number (-32),

$$\mathbf{A_{2,3}} = \begin{pmatrix} 9 & \frac{9}{2} & \frac{9}{2} & \frac{9}{4} \\ \frac{9}{2} & 9 & \frac{9}{4} & \frac{9}{2} \\ \frac{9}{2} & \frac{9}{4} & 9 & \frac{9}{2} \\ \frac{9}{4} & \frac{9}{2} & \frac{9}{2} & 9 \end{pmatrix}$$

and, in general, if

$$\mathbf{A_0} = \mathbf{I_{s-1}} + \mathbf{One_{s-1}}$$

then $\mathbf{A'_s}$ satisfies the following identity (each row and column is formed by $s-1$ blocks):

$$\mathbf{A'_s} = \begin{pmatrix} 2\mathbf{A_0} & \mathbf{A_0} & \dots & \mathbf{A_0} \\ \mathbf{A_0} & 2\mathbf{A_0} & \dots & \mathbf{A_0} \\ \vdots & & \dots & \\ \mathbf{A_0} & \mathbf{A_0} & \dots & 2\mathbf{A_0} \end{pmatrix}.$$

As a consequence of the identity (5.44) the eigenvalues of $\mathbf{A_{r,s}}$ are equal to $X_{r,s}$ times the eigenvalues of $\mathbf{A'_s}$, and therefore the value of (5.28) can be computed once we know the spectrum of $\mathbf{A_s'}$. The next Theorem provides the relevant information about $\mathbf{A_s'}$. In its proof we will repeatedly use the following well-known fact:

Lemma 5.14 *Let N be a real number. Let $\mathbf{M}$ be a real symmetric matrix with fixed row sum N. Then N is an eigenvalue of $\mathbf{M}$.*

Theorem 5.15 *For any integer $s \geq 2$,*

$$\mathrm{Spec}\,\mathbf{A_s'} = \begin{pmatrix} 1 & s & s^2 \\ (s-2)^2 & 2(s-2) & 1 \end{pmatrix}.$$

Proof. To prove the result we will define $(s-1)^2$ linearly independent eigenvectors of $\mathbf{A_s'}$ and retrieve the corresponding eigenvalues. Note that all eigenvectors corresponding to different eigenvalues are linearly independent, (see, for example, [62, Theorem 8.2, p. 186]) and so we will only prove linear independence for sets of eigenvectors sharing the same eigenvalue.

We first claim that s^2 is an eigenvalue of $\mathbf{A_s'}$ with multiplicity one. To this end note that each row of $\mathbf{A_s'}$ has one element equal to four, $2(s-2)$ elements equal to two, and the remaining $(s-2)^2$ elements equal to one. Thus the sum of the elements in each row is s^2 and the claim now follows from Lemma 5.14.

Next we argue that $\mathbf{A_s'}$ has $(s-2)^2$ linearly independent eigenvectors with corresponding eigenvalue equal to one. Let i and j be positive integers in the range $\{2, \ldots, s-1\}$. Define the (column) vectors $\mathbf{v^{i,j}} \in \mathbb{Z}^{(s-1)^2}$ as follows:

$$(\mathbf{v^{i,j}})_\imath = \begin{cases} 1 & \imath = (1,1) \quad \text{or} \quad \imath = (i,j) \\ -1 & \imath = (i,1) \quad \text{or} \quad \imath = (1,j) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the resulting $(s-2)^2$ vectors are linearly independent since the 1 in row (i,j) is unique to each of them. We now argue that

$$\mathbf{A}'_{\mathbf{s}}\mathbf{v}^{\mathbf{i},\mathbf{j}} = \mathbf{v}^{\mathbf{i},\mathbf{j}}.$$

To believe this, note that the first element of $\mathbf{A}'_{\mathbf{s}}\mathbf{v}^{\mathbf{i},\mathbf{j}}$ (indexed by the pair $(1,1)$) is always equal to one as, by definition, $\mathbf{v}^{\mathbf{i},\mathbf{j}}$ will add together $(\mathbf{A}'_{\mathbf{s}})_{(1,1),(1,1)} = 4$, $(\mathbf{A}'_{\mathbf{s}})_{(1,1),(1,j)} = 2 = (\mathbf{A}'_{\mathbf{s}})_{(1,1),(i,1)}$ (with negative sign) and $(\mathbf{A}'_{\mathbf{s}})_{(1,1),(i,j)} = 1$. The same is true for the element indexed by the pair (i,j). The element of index $(1,j)$ is equal to minus one as it is formed by adding $(\mathbf{A}'_{\mathbf{s}})_{(1,j),(1,1)} = 2$, $(\mathbf{A}'_{\mathbf{s}})_{(1,j),(i,1)} = 2$, $(\mathbf{A}'_{\mathbf{s}})_{(1,j),(1,j)} = 4$ (with the negative sign as $(\mathbf{v}^{\mathbf{i},\mathbf{j}})_{(1,j)} = -1$) and $(\mathbf{A}'_{\mathbf{s}})_{(1,j),(i,j)} = 1$. A similar argument applies to the element of index $(i,1)$. Finally any other element of $\mathbf{A}'_{\mathbf{s}}\mathbf{v}^{\mathbf{i},\mathbf{j}}$ is zero as the contributions from $\mathbf{A}'_{\mathbf{s}}$ cancel out.

Finally we show that $\mathbf{A}'_{\mathbf{s}}$ has $2(s-2)$ linearly independent eigenvectors with eigenvalue s. First let i be a positive integer in $\{2,\ldots,s-2\}$ and define $\mathbf{v}^{\mathbf{i}} \in \mathbb{Z}^{(s-1)^2}$ as follows:

$$(\mathbf{v}^{\mathbf{i}})_{\imath} = \begin{cases} 1 & \imath = (1,j), j \in \{1,\ldots,s-1\} \\ -1 & \imath = (i,j), j \in \{1,\ldots,s-1\} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(\mathbf{v}^{\mathbf{i}})^T = (\mathbf{1}_{\mathbf{s-1}}, \mathbf{0}_{\mathbf{s-1}}, \ldots, \mathbf{0}_{\mathbf{s-1}}, -\mathbf{1}_{\mathbf{s-1}}, \mathbf{0}_{\mathbf{s-1}}, \ldots, \mathbf{0}_{\mathbf{s-1}})$$

(the first block is always $\mathbf{1}_{\mathbf{s-1}}$ and the only other non-zero block equals $-\mathbf{1}_{\mathbf{s-1}}$ and is in position i). As before, notice that the resulting $s-3$ vectors are linearly independent as the $-\mathbf{1_{s-1}}$ block is unique to each vector. Furthermore, the j^{th} element in the product $\mathbf{A'_s v^i}$ is equal to s if $j \in \{1, \ldots, s-1\}$ (this follows essentially by Lemma 5.14, as $\mathbf{A_0}$ has fixed row sum equal to s), $-s$ if j corresponds to $\imath$ of the form (i, l) for some $l \in \{1, \ldots, s-1\}$ and zero otherwise. Therefore we have

$$\mathbf{A'_s v^i} = s\mathbf{v^i}.$$

To complete the definition of the spectrum of $\mathbf{A'_s}$, consider the vectors $\mathbf{w^i}$ for $i \in \{1, \ldots, s-1\}$ defined as follows:

$$(\mathbf{w^i})_\imath = \begin{cases} 1 & \imath = (1, j), j \in \{1, \ldots, s-1\} \setminus i \\ -1 & \imath = (j, i), j \in \{2, \ldots, s-1\} \\ 0 & \text{otherwise.} \end{cases}$$

The resulting $s-1$ vectors are linearly independent, they are also linearly independent from the $\mathbf{v^i}$ eigenvectors as $(\mathbf{w^i})_{\{s-1,i\}} = -1$, but $(\mathbf{v^l})_{\{s-1,i\}} = 0$ for all l. We claim that

$$\mathbf{A'_s w^i} = s\mathbf{w^i}.$$

Table 5.3 gives an example for $s = 4$. In general, to prove this we look at $(\mathbf{A'_s})_\imath \mathbf{w^i}$, the scalar product of the row indexed by $\imath$ of $\mathbf{A'_s}$ and $\mathbf{w^i}$. Several cases arise. We will describe explicitly only the case $j \neq i$. The case $j = i$ can be analysed similarly. If $\imath = (1, j)$ then the sum of the first $s-1$ terms is $(2s-2)$ (the 2 due to the fact that $(\mathbf{w^i})_{(1,i)} = 0$), and the final result is s because of the contribution from the terms multiplying the elements of $\mathbf{w^i}$ equal to one. If $\imath = (l, j)$, for $l \geq 2$ then the sum of the first $s-1$ terms is always $(s-1)$ (the 1, as before, due to the fact that $(\mathbf{w^i})_{(1,i)} = 0$) and the final result is zero because $\mathbf{w^i}$ has $s-2$ components equal to one, one of

$$\begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 2 & 4 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 & 4 & 2 & 2 \\ 1 & 2 & 1 & 1 & 2 & 1 & 2 & 4 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \\ -4 \\ 0 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

Table 5.3: The expression above shows the product $\mathbf{A_4'w^2}$ and its result. It is easy to verify that $\mathbf{A_4'w^2} = 4\mathbf{w^2}$.

them gets multiplied by two (the value of $(\mathbf{A_s'})_{(l,j),(l,i)})$ and all the others are multiplied by one. ■

The asymptotic expression for $\Phi_{r,s}(n)$ used in the statement of Theorem 5.11 can now be readily obtained by combining the expression for $T_{r,2}\left(\overline{\mathbf{m}}\right)$ given in Lemma 5.12, using Lemma 5.3 which provides an expression for $\kappa(B_{s,2})$, and

$$s^{1-s}\left(\frac{2\pi n}{X_{r,s}}\right)^{\frac{(s-1)^2}{2}}$$

an expression derived from (5.28) using the argument about the eigenvalues of $\mathbf{A_{r,s}}$ and Theorem 5.15.

5.3.2 Approximating the Tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$

In this section we complete the proof of Theorem 5.11. The following result gives the required estimate on the tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$. Recall from equation (5.27) that for each integer $r \geq 2$ and $s \geq 2$, $a_n = n^{-\frac{s-1}{2}}(c_{r,s})^n$, where $c_{r,s} = s^{\frac{1}{r}-1}(s-1)$.

Theorem 5.16 *Let n, s and r be positive constant integers with $r \geq 2$, $s \geq 3$ and*

$$r < \frac{s}{2}\log(s-1).$$

Let $\rho(n)$ be a function such that

$$\lim_{n\to\infty} \frac{\rho(n)}{n^{1/2}} = \infty.$$

Then

$$\sum_{\mathbf{m}:||\mathbf{m}-\overline{\mathbf{m}}||_2 \geq \rho(n)} T_{r,2}(\mathbf{m}) = o(a_n^2) \tag{5.45}$$

as n tends to infinity.

The crucial point in the proof of Theorem 5.16 will be the definition of an upper bound for the tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ (see Lemma 5.21 below) that allows us to exploit, at least partially, the fact that the s^2-compositions we work with are balanced. It is not difficult to relate the sum in (5.45) to a standard multinomial sum, and then use well-known techniques (dating back to Hoeffding [45]) to bound this using large deviation inequalities. However this leads to results that are too weak. To get a bound on sum in (5.45) that suffices to prove Theorem 5.11 we will use the fact that $\sum_{j=1}^{s} m_{(i,j)} = \frac{n}{sr}$ for each $i \in \{1,\ldots,s\}$. The whole sum will then be factorised into a number of parts which only depend on s elements of $\mathbf{m}$. Studying such components and their variations as $\mathbf{m}$ moves away from $\overline{\mathbf{m}}$ will enable us to complete our proof.

Before moving to the main result of this section we need to highlight a few properties of three functions that will be important for our reasoning. In what follows n, s and r are positive integers, s and r are greater than one, and the three numbers are such that $\frac{n}{s^2r}$ is an integer. Let m be a positive integer in the range $\left\{\frac{n}{s^2r}, \ldots, \frac{n}{sr}\right\}$, and for $z \in \left\{\frac{m}{2}, \ldots, m-1\right\}$ define $h_{n,m}(z)$ as

$$\frac{((s-2)n + srz + sr)^{rz}((s-2)n + sr(m-z) + sr)^{r(m-z)}}{z^{z+1/2}(m-z)^{m-z+1/2}}.$$

Also, define the following functions of s-compositions of $\frac{n}{sr}$:

$$H'_n(\mathbf{m}) = \binom{n/sr}{m_1, \ldots, m_s} \prod_{i:m_i \neq 0} \left(\frac{s-2}{s} + \frac{r(m_i+1)}{n}\right)^{rm_i}$$

and

$$H_n(\mathbf{m}) = \left(\frac{n}{sr}\right)! \frac{e^{\frac{n}{sr}}}{(2\pi)^{\frac{s}{2}}} \prod_{i:m_i \neq 0} \frac{\left(\frac{s-2}{s} + \frac{r(m_i+1)}{n}\right)^{rm_i}}{m_i^{m_i+1/2}}.$$

We start our investigation by showing that $H_n(\mathbf{m})$ can be bounded above by a function that only depends on the square of the Euclidean distance between $\mathbf{m}$ and $\overline{\mathbf{m}}$ (in what follows $\mathbf{x} = \mathbf{m} - \overline{\mathbf{m}}$). It should be remarked that here and in the rest of this section, unless otherwise stated, $\overline{\mathbf{m}}$ has s components, each equal to $\frac{n}{s^2r}$. We keep the same notation we used at the beginning of Section 5.3 to refer to a vector of size s^2 because the two vectors play similar roles.

Lemma 5.17 *Let n, s and r be positive constant integers with $r \geq 2$, and $s \geq 3$, and let $\mathbf{m}$ be an s-composition of $\frac{n}{sr}$ with $||\mathbf{x}||_2 = o(n)$ as n tends to* *infinity. Then $H_n(\mathbf{m})$ can be bounded above by a function that approaches*

$$\left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}}\left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}}\right)^{\frac{n}{s}}\frac{s^{\frac{5s}{2}}}{(s-1)^{2s}}\mathrm{e}^{\left(\frac{s^2r^2}{(s-1)^2n}-\frac{s^2r}{2n}\right)||\mathbf{x}||_2{}^2+\frac{r}{(s-1)^2}-\frac{s^2r}{(s-1)^2n}+\frac{sr}{12n}}$$

as n tends to infinity.

Proof. We first notice that

$$\frac{\left(\frac{n}{sr}\right)!\mathrm{e}^{\frac{n}{sr}}}{(2\pi)^{\frac{s}{2}}\prod_{i:m_i\neq 0} m_i^{m_i+1/2}}, \tag{5.46}$$

can be written as

$$\frac{\left(\frac{n}{sr}\right)!\mathrm{e}^{\frac{n}{sr}}}{(2\pi)^{\frac{s}{2}}\prod_{i=1}^{s}\frac{n}{s^2r}^{\frac{n}{s^2r}+\frac{1}{2}}}\prod_{i=1}^{s}\left(\frac{n}{s^2rm_i}\right)^{m_i+1/2}$$

(in particular note that, because $||\mathbf{x}||_2 = o(n)$, no m_i can be zero). Using Stirling's approximation as in (5.24) we can bound this above by

$$\frac{\left(\frac{n}{sr}\right)^{\frac{n}{sr}+\frac{1}{2}}\mathrm{e}^{\frac{sr}{12n}}}{(2\pi)^{\frac{s-1}{2}}\frac{n}{s^2r}^{\frac{n}{sr}+\frac{s}{2}}}\prod_{i=1}^{s}\left(\frac{n}{s^2rm_i}\right)^{m_i+1/2} = \left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}}s^{\frac{n}{sr}+\frac{s}{2}}\mathrm{e}^{\frac{sr}{12n}}\prod_{i=1}^{s}\left(\frac{n}{s^2rm_i}\right)^{m_i+1/2} \tag{5.47}$$

We now find an upper bound on the product in (5.47).

$$\begin{aligned}\prod_{i=1}^{s}\left(\frac{n}{s^2rm_i}\right)^{m_i+1/2} &= \prod_{i=1}^{s}\left(1+\frac{s^2rx_i}{n}\right)^{-(m_i+1/2)}\\ &\leq \prod_{i=1}^{s}\mathrm{e}^{-x_i-\frac{s^2r}{n}x_i^2-\frac{s^2r}{2n}x_i}\\ &\leq \mathrm{e}^{-\frac{s^2r}{2n}||\mathbf{x}||_2{}^2},\end{aligned} \tag{5.48}$$

where the second line follows from Lemma 2.1 and the third one uses the fact that $\sum_i x_i = 0$. We can now bound (5.46) above by

$$\left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}} s^{\frac{n}{sr}+\frac{s}{2}} e^{\frac{sr}{12n}-\frac{s^2r}{2n}||\mathbf{x}||_2{}^2}. \tag{5.49}$$

We now look at the product in $H_n(\mathbf{m})$. Remembering that $m_i = \frac{n}{s^2r} + x_i$, and noting that as $||\mathbf{x}||_2 = o(n)$, no m_i may be equal to zero. Thus

$$\prod_{i:m_i\neq 0}\left(\frac{s-2}{s}+\frac{r(m_i+1)}{n}\right)^{rm_i}$$

is equal to

$$\prod_{i=1}^{s}\left(\frac{(s-1)^2}{s^2}+\frac{r(x_i+1)}{n}\right)^{rm_i} = \left(\frac{s-1}{s}\right)^{\frac{2n}{s}-2s}\prod_{i=1}^{s}\left(1+\frac{s^2r(x_i+1)}{(s-1)^2n}\right)^{rm_i}.$$

Using Lemma 2.1 we can bound this above by

$$\left(\frac{s-1}{s}\right)^{\frac{2n}{s}-2s}\prod_{i=1}^{s} e^{\left(\frac{r}{(s-1)^2}-\frac{s^2r}{(s-1)^2n}\right)(x_i+1)+\frac{s^2r^2}{(s-1)^2n}x_i(x_i+1)},$$

which, remembering that $\sum_i x_i = 0$ is equal to

$$\left(\frac{s-1}{s}\right)^{\frac{2n}{s}-2s} e^{\frac{s^2r^2}{(s-1)^2n}||\mathbf{x}||_2{}^2+\frac{r}{(s-1)^2}-\frac{s^2r}{(s-1)^2n}}. \tag{5.50}$$

The result follows by multiplying together equations (5.49) and (5.50). ∎

In the proof of Theorem 5.16 we will need to argue that $H'_n(\mathbf{m})$ is a decreasing function of the Euclidean distance between $\mathbf{m}$ and $\overline{\mathbf{m}}$. This does not seem easy. However we are able to prove that $H'_n(\mathbf{m})$ is asymptotically not too far from $H_n(\mathbf{m})$ (see Lemma 5.18 below), that the way in which $H_n(\mathbf{m})$ varies as the Euclidean distance $||\mathbf{m}-\overline{\mathbf{m}}||_2$ increases is completely described by the trend properties of $h_{n,m}(z)$, and finally that the latter has the right monotonicity, implying that $H_n(\mathbf{m})$ is indeed a decreasing function of $||\mathbf{m} - \overline{\mathbf{m}}||_2$. We start with a result relating $H_n'(\mathbf{m})$ and $H_n(\mathbf{m})$.

Lemma 5.18 *Let n, s and r be positive constant integers with $r \geq 2$, and $s \geq 4$, and let $\mathbf{m}$ be an s-composition of $\frac{n}{sr}$. Then*

$$\mathrm{e}^{-\sum_{i:m_i\neq 0}\frac{1}{12m_i}} \leq \frac{H_n'(\mathbf{m})}{H_n(\mathbf{m})} \leq 1$$

for n sufficiently large.

Proof. $H_n'(\mathbf{m})$ divided by $H_n(\mathbf{m})$ is equal to

$$\prod_{i:m_i\neq 0} \frac{\sqrt{2\pi} m_i^{m_i+1/2}\mathrm{e}^{-m_i}}{m_i!}. \tag{5.51}$$

Using the bounds on $n!$ in (5.24) we can sandwich the product in (5.51) between $\mathrm{e}^{-\sum_{i:m_i\neq 0}\frac{1}{12m_i}}$ and one. ∎

Next we analyse the way in which $h_{n,m}(z)$ varies.

Lemma 5.19 *Let n, s and r be positive constant integers with $r \geq 2$, $s \geq 3$ and*

$$r < \frac{s}{2}\log(s-1).$$

Let m be an integer number with $\frac{n}{s^2r} \leq m \leq \frac{n}{sr}$. Then, for n sufficiently large, $h_{n,m}(z)$ is decreasing for $z \in \left\{\max\left\{\frac{m}{2}, m - \frac{n}{s^2r}\right\}, \ldots, m-1\right\}$.

Proof. We will prove that $h_{n,m}(z)$ is decreasing as a function of the real variable $z \in \left[\frac{m}{2}, m\right)$. Since $h_{n,m}(z)$ is defined, continuous and strictly positive for all z in this range, $h_{n,m}(z)$ is increasing if and only if $\log(h_{n,m}(z))$ is. The first derivative of $\log(h_{n,m}(z))$ is

$$r\log\left(1 + \frac{sr(2z-m)}{n(s-2) + sr(m-z) + sr}\right) + \log\left(\frac{m-z}{z}\right)$$

$$+\frac{r^2 s(2z-m)(n(s-2)+sr)}{(rsz+n(s-2)+sr)(rs(m-z)+n(s-2)+sr)}+\frac{2z-m}{2z(m-z)} \tag{5.52}$$

This has a stationary point when $z = \frac{m}{2}$.

We will now show that if $r < \frac{s}{2}\log(s-1)$, the derivative of $\log(h_{n,m}(z))$ is negative for all $z > \max\left\{\frac{m}{2}, m - \frac{n}{s^2 r}\right\}$, and thus $h_{n,m}(z)$ is monotonically decreasing. Notice that

$$\log\left(\frac{m-z}{z}\right) \tag{5.53}$$

is negative when $z > \frac{m}{2}$ and that

$$r\log\left(1+\frac{sr(2z-m)}{n(s-2)+sr(m-z)+sr}\right), \tag{5.54}$$

$$\frac{r^2 s(2z-m)(n(s-2)+sr)}{(rsz+n(s-2)+sr)(rs(m-z)+n(s-2)+sr)} \tag{5.55}$$

and

$$\frac{2z-m}{2z(m-z)} \tag{5.56}$$

are positive. By Lemma 2.1, for $z \geq \frac{m}{2}$, (5.54) can be bounded above by

$$\frac{sr^2(2z-m)}{n(s-2)+sr(m-z)+sr}. \tag{5.57}$$

We can now add equations (5.55), (5.56) and (5.57) together giving

$$\begin{aligned} &(2z-m)\left(\frac{sr^2}{rs(m-z)+n(s-2)+sr}\left(1+\frac{n(s-2)+sr}{rsz+n(s-2)+sr}\right)+\frac{1}{2z(m-z)}\right) \\ \leq\ &(2z-m)\left(\frac{2sr^2}{n(s-2)}+\frac{1}{2z(m-z)}\right). \end{aligned} \tag{5.58}$$

We will now split the domain into two parts, the first where $z \geq Cm$ for $1/2 < C < 1$ sufficiently large such that the magnitude of (5.53) is much larger than (5.58) and the second where $z < Cm$ and so $\frac{1}{2z(m-z)} < \frac{1}{C(1-C)m^2}$.

Expression (5.58) is maximised when $z = m - 1$ and hence can be bounded above by

$$(m-2)\left(\frac{2sr^2}{n(s-2)} + \frac{1}{2(m-1)}\right) \leq \frac{2sr^2m}{n(s-2)} + \frac{m-2}{2m-2} < \frac{2sr^2m}{n(s-2)} + \frac{1}{2},$$

noticing also that $m \leq \frac{n}{sr}$, we can bound this above by

$$\frac{2r}{s-2} + \frac{1}{2}. \tag{5.59}$$

Remembering that $r < \frac{s}{2}\log(s-1)$ we can bound (5.59) above by

$$\frac{s\log(s-1)}{s-2} + \frac{1}{2}. \tag{5.60}$$

When $z \geq Cm$, we can also bound (5.53) above by

$$\log\left(\frac{1-C}{C}\right), \tag{5.61}$$

and when

$$C \geq \frac{(s-1)^{\frac{s}{s-2}}\mathrm{e}^{1/2}}{(s-1)^{\frac{s}{s-2}}\mathrm{e}^{1/2}+1},$$

the sum of (5.59) and (5.61) is negative. Hence for all $z \geq Cm$, $h'_{n,m}(z)$ is negative.

If $\max\left\{\frac{m}{2}, m - \frac{n}{s^2r}\right\} \leq z \leq Cm$, then

$$\frac{1}{2z(m-z)} \leq \frac{1}{2C(1-C)m^2},$$

and so we can give an upper bound on (5.58) of

$$(2z-m)\left(\frac{2sr^2}{n(s-2)} + \frac{1}{2C(1-C)m^2}\right) \tag{5.62}$$

We can now use equations (5.53) and (5.62) to give an upper bound on $\frac{d}{dz}\log(h_{n,m}(z))$ of $l_{n,m}(z)$, where $l_{n,m}(z)$ is defined as

$$l_{n,m}(z) = \log\left(\frac{m-z}{z}\right) + (2z-m)\left(\frac{2sr^2}{n(s-2)} + \frac{1}{2C(1-C)m^2}\right).$$

We take the first derivative of $l_{n,m}(z)$, giving

$$\frac{4sr^2z(m-z) - mn(s-2)}{z(m-z)n(s-2)} + \frac{1}{C(1-C)m^2}. \tag{5.63}$$

Since $z \geq \max\left\{\frac{m}{2}, m - \frac{n}{s^2r}\right\}$, we will consider two cases, the first where $\frac{n}{s^2r} \leq m \leq \frac{2n}{s^2r}$ and $z^* = \frac{m}{2}$, and the second where $\frac{2n}{s^2r} \leq m \leq \frac{n}{sr}$ and $z^* = m - \frac{n}{s^2r}$. In each case we will show that $l_{n,m}(z^*) \leq 0$ for all sufficiently large n, and that $l'_{n,m}(z) < 0$ for all $z^* \leq z \leq Cm$ and sufficiently large n.

When $\frac{n}{s^2r} \leq m \leq \frac{2n}{s^2r}$ and $z^* = \frac{m}{2}$, $l_{n,m}(z^*)$ is equal to zero, when $\frac{2n}{s^2r} \leq m \leq \frac{n}{sr}$ and $z^* = m - \frac{n}{s^2r}$, $l_{n,m}(z^*)$ is at most

$$\begin{aligned}\log\left(\frac{\frac{n}{s^2r}}{\frac{(s-1)n}{s^2r}}\right) + \left(\frac{(s-2)n}{s^2r}\right)\left(\frac{2sr^2}{n(s-2)} + \frac{1}{2C(1-C)m^2}\right) &= \\ = \log\left(\frac{1}{s-1}\right) + \frac{2r}{s} + \frac{(s-2)r}{2C(1-C)n},\end{aligned}$$

which is negative for $r < \frac{s}{2}\log(s-1)$.

We now consider the first derivative of $l_{n,m}(z)$ given in equation (5.63), this is negative if and only if for all z such that $z^* \leq z \leq Cm$,

$$4sr^2z(m-z) - mn(s-2) < 0, \tag{5.64}$$

and

$$\left|\frac{4sr^2z(m-z) - mn(s-2)}{z(m-z)n(s-2)}\right| > \frac{1}{C(1-C)m^2}. \tag{5.65}$$

When $\frac{n}{s^2r} \leq m \leq \frac{2n}{s^2r}$ and $z^* = \frac{m}{2}$, the left hand side of (5.64) is at most

$$\begin{aligned} & 4sr^2\frac{m^2}{4} - mn(s-2) \\ = \;& m(srm - n(s-2)) \\ \leq \;& nm\left(\frac{2}{s} - (s-2)\right), \end{aligned} \tag{5.66}$$

which is negative for all $s \geq 3$.

When $\frac{2n}{s^2r} \leq m \leq \frac{n}{sr}$ and $z^* = m - \frac{n}{s^2r}$, the left hand side of (5.64) is at most

$$\begin{aligned} & 4sr^2\frac{n}{s^2r}\left(m - \frac{n}{s^2r}\right) - mn(s-2) \\ = \;& n\left(\frac{4r}{s}\left(m - \frac{n}{s^2r}\right) - m(s-2)\right) \\ = \;& n\left(m\left(\frac{4r}{s} - (s-2)\right) - \frac{4n}{s^3}\right). \end{aligned} \tag{5.67}$$

If

$$4r \leq s(s-2),$$

then $\frac{4r}{s} - (s-2) \leq 0$ and so (5.67) is negative, otherwise (5.67) can be bounded above by the case where $m = \frac{n}{sr}$, and so the inequality holds if

$$\begin{aligned} \frac{n}{sr}\left(\frac{4r}{s} - (s-2)\right) &< \frac{4n}{s^3} \\ \left(\frac{4r}{s} - (s-2)\right) &< \frac{4r}{s^2} \\ 4r(s-1) &< s^2(s-2). \end{aligned} \tag{5.68}$$

Since

$$r < \frac{s}{2}\log(s-1),$$

equation (5.68) holds whenever

$$2(s-1)\log(s-1) < s(s-2),$$

which is true for all $s \geq 3$.

Finally, we need to show that inequality (5.65) holds for all m and z in the range being considered. From (5.66), (5.67) and the fact that $\frac{n}{s^2r} \leq m \leq \frac{n}{sr}$ and $\max\left\{\frac{m}{2}, m - \frac{n}{s^2r}\right\} \leq z \leq Cm$, we can see that the left hand side of inequality (5.65) is at least

$$\frac{C_1}{n},$$

for some positive constant C_1. For sufficiently large n it is therefore larger than the right hand side, and the inequality holds.

Since $l_{n,m}(z^*) < 0$, and $l'_{n,m}(z) < 0$ for all $z^* \leq z \leq Cm$, it follows that $l_{n,m}(z) < 0$ for all $z > z^*$. As $l_{n,m}(z)$ is an upper bound on $\frac{d}{dz}\log(h_{n,m}(z))$, it therefore follows that for any $z > z^*$, $s \geq 4$, and r such that

$$r < \frac{s}{2}\log(s-1),$$

$\frac{d}{dz}\log(h_{n,m}(z))$ is negative and so $h_{n,m}(z)$ is decreasing. ∎

We can now complete the investigation of the properties of functions $h_{n,m}(z)$, $H_n(\mathbf{m})$, and $H'_n(\mathbf{m})$ by showing that, under certain circumstances, $H_n(\mathbf{m})$ is a decreasing function of the Euclidean distance between $\mathbf{m}$ and $\overline{\mathbf{m}}$. The result will follow from the fact that the growth of $H_n(\mathbf{m})$ is in fact described accurately by that of $h_{n,m}(z)$.

Lemma 5.20 *Let n, s and r be positive constant integers with $r \geq 2$, $s \geq 3$ and*

$$r < \frac{s}{2}\log(s-1),$$

and let $\mathbf{m}$ *be an s-composition of* $\frac{n}{sr}$ *with* $||\mathbf{x}||_2 > 0$. *Then for sufficiently large* n, *there exists some* $\mathbf{m}'$ *with* $||\mathbf{x}'||_2 < ||\mathbf{x}||_2$ *such that*

$$H_n(\mathbf{m}') > H_n(\mathbf{m}).$$

Proof. Let m_i and m_j be elements of $\mathbf{m}$ with $m_i = \max_l m_l$, $m_j = \min_l m_l$ and $m_i \geq m_j + 2$, and let $\mathbf{m}'$ be defined such that $m_i' = m_i - 1$, $m_j' = m_j + 1$ and $m_l' = m_l$ for all $1 \leq l \leq s$, $l \notin \{i, j\}$. Firstly we will consider the case where $m_j = 0$. In this case

$$\frac{H_n(\mathbf{m}')}{H_n(\mathbf{m})}$$

is equal to

$$\begin{aligned}
\frac{\left(\frac{s-2}{s} + \frac{rm_i}{n}\right)^{rm_i'}}{\left(\frac{s-2}{s} + \frac{r(m_i+1)}{n}\right)^{rm_i}} \times \frac{m_i^{m_i+1/2}}{(m_i')^{m_i'+1/2}} \times \left(\frac{s-2}{s} + \frac{r}{n}\right)^{r-1} \frac{n}{r} = \\
= \left(1 - \frac{srn}{(s-2)n + sr(m_i+1)}\right)^{r(m_i-1)} \left(\frac{s-2}{s} + \frac{rm_i}{n}\right)^{r} \\
\times \left(1 + \frac{1}{m_i - 1}\right)^{m_i - 1/2} m_i \left(\frac{s-2}{s} + \frac{r}{n}\right)^{r-1} \frac{n}{r} \\
\geq Cn,
\end{aligned}$$

for some positive constant C. It therefore follows that if n is large enough, $H_n(\mathbf{m}') > H_n(\mathbf{m})$.

When $m_j > 0$, we can easily see that

$$H_n(\mathbf{m}') = H_n(\mathbf{m}) \frac{h_{n,m_i+m_j}(m_i')}{h_{n,m_i+m_j}(m_i)}.$$

Note that as m_j is the minimum of all m_l, it must be the case that $m_j < \frac{n}{s^2 r}$ and hence $m_i > (m_i + m_j) - \frac{n}{s^2 r}$, therefore by Lemma 5.19, when $s \geq 3$ and

$r < \frac{s}{2}\log(s-1)$

$$h_{n,m_i+m_j}(m_i') > h_{n,m_i+m_j}(m_i).$$

■

The next lemma formalises the way in which the tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$ is bounded above by a quantity that allows us to exploit (at least partially) the fact that the s^2-compositions of n/r we are working with are balanced. As usual, in the following statement, if $\mathbf{m}$ is a tuple of t non-negative integers summing to some value N, then $\mathbf{x}$ is the tuple having

$$x_i = m_i - \frac{N}{t}$$

for each $i \in \{1, \ldots, t\}$. The tuple $\mathbf{x}$ quantifies the displacement of each component of $\mathbf{m}$ from its arithmetic mean.

Lemma 5.21 *Let n, s and r be positive integers such that $\frac{n}{s^2 r}$ is an integer, and let R be a positive real number. Let $S_k \subset \mathbb{Z}^{s^k}$, for $k = 1, 2$, be a set of (resp. balanced) s-compositions (resp. s^2-compositions) of $\frac{n}{s^{2-k} r}$. Then*

$$\sum_{\mathbf{x}\in S_2:||\mathbf{x}||_2\geq R} T_{r,2}(m_{1(1)}, \ldots, m_{1(s^2)}) \leq$$

$$C\binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \sum_{h=1}^{s} \binom{s}{h} \left(\sum_{\mathbf{x}\in S_1:||\mathbf{x}||_2\geq \frac{R}{\sqrt{s}}} H_n'(\mathbf{m}) \right)^h \left(\sum_{\mathbf{x}\in S_1:||\mathbf{x}||_2< \frac{R}{\sqrt{s}}} H_n'(\mathbf{m}) \right)^{s-h},$$

for some positive constant C.

Proof. From equation (5.15), $T_{r,2}$ is equal to

$$\binom{n/r}{m_{1(1)}, \ldots, m_{1(s^2)}} \prod_{\{1:m_1\neq 0\}} \left(\frac{rd_{B_{s,2}}(1) + r}{n}\right)^{m_1(r-1)} \left(\frac{rd_{B_{s,2}}(1)}{n}\right)^{m_1 - 1} \times$$

$$\sum_{T} \prod_{\{1:m_1\neq 0\}} \left(\frac{rm_1}{n}\right)^{\deg_T(1)-1}$$

We first look at the sum

$$\sum_{T} \prod_{\{1:m_1\neq 0\}} \left(\frac{rm_1}{n}\right)^{\deg_T(1)-1}. \tag{5.69}$$

The terms within this sum are maximised when there are s non-zero m_1, each of which is equal to $\frac{n}{sr}$, using this as an upper bound the sum is at most $\left(\frac{1}{s}\right)^{s-2}$ multiplied by the number of spanning trees of $B_{s,2}$. By Lemma 5.3, $\kappa(B_{s,2})$ is equal to

$$s^{2s-4}(s-1)^{2(s-1)} + s^{(s-1)^2-2}(s-2)^{(s-1)^2}$$

and so equation (5.69) is at most

$$\begin{aligned} & \left(\frac{1}{s}\right)^{s-2} \left(s^{2s-4}(s-1)^{2(s-1)} + s^{(s-1)^2-2}(s-2)^{(s-1)^2}\right) \\ = \; & s^{s-2}\left((s-1)^{2(s-1)} + s^{(s-1)(s-3)}(s-2)^{(s-1)^2}\right). \end{aligned}$$

We can therefore bound $T_{r,2}(m_{1(1)}, \ldots, m_{1(s^2)})$ above by

$$C\binom{n/r}{m_{1(1)}, \ldots, m_{1(s^2)}} \prod_{\{1:m_1\neq 0\}} \left(\frac{rd_{B_{s,2}}(1)+r}{n}\right)^{m_1(r-1)} \left(\frac{rd_{B_{s,2}}(1)}{n}\right)^{m_1-1}, \tag{5.70}$$

where

$$C = s^{s-2}\left((s-1)^{2(s-1)} + s^{(s-1)(s-3)}(s-2)^{(s-1)^2}\right).$$

Equation (5.70) can then be bounded above by

$$C\binom{n/r}{m_{1(1)}, \ldots, m_{1(s^2)}} \prod \left(\frac{d_{B_{s,2}}(1)+1}{n/r}\right)^{rm_1}.$$

Next, we notice that

$$\binom{n/r}{m_{1(1)},\ldots,m_{1(s^2)}} = \binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}} \prod_{j=0}^{s-1} \binom{n/sr}{m_{1(sj+1)},\ldots,m_{1(sj+s)}}.$$

This suggests the following bound, obtained by splitting the factors of the product defining $T_{r,2}(m_{1(1)},\ldots,m_{1(s^2)})$ according to different values of j in the multinomial product above:

$$C \sum_{\mathbf{x}\in S_2:||\mathbf{x}||_2\geq R} T_{r,2}(m_{1(1)},\ldots,m_{1(s^2)}) \leq \binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}} \times$$
$$\sum_{m_{1(1)},\ldots,m_{1(s)}} \binom{n/sr}{m_{1(1)},\ldots,m_{1(s)}} \prod_{i=1}^{s} \left(\frac{rd_{B_{s,2}}(\imath(i))+r}{n}\right)^{rm_{1(i)}} \times$$
$$\sum_{m_{1(s+1)},\ldots,m_{1(2s)}} \binom{n/sr}{m_{1(s+1)},\ldots,m_{1(2s)}} \prod_{i=1}^{s} \left(\frac{rd_{B_{s,2}}(\imath(s+i))+r}{n}\right)^{rm_{1(s+i)}} \times$$
$$\ldots\times$$
$$\sum_{m_{1(s^2-s+1)},\ldots,m_{1(s^2)}} \binom{n/sr}{m_{1(s^2-s+1)},\ldots,m_{1(s^2)}} \prod_{i=1}^{s} \left(\frac{rd_{B_{s,2}}(\imath(s^2-s+i))+r}{n}\right)^{rm_{1(s^2-s+i)}} \tag{5.71}$$

Here each sum is over a number of s-compositions of n/sr and the distance between the whole $(m_{1(1)},\ldots,m_{1(s^2)})$ and $\overline{\mathbf{m}} \in \mathbb{Z}^{s^2}$ is at least R. Of course, by using (5.3) to rewrite $d_{B_{s,2}}(\imath(sj+i))$, we can easily see that each sum in (5.71) involves a term of the form $H'_n(\mathbf{m})$.

The argument is completed by noticing that

$$||\mathbf{x}||_2 = ||\mathbf{m}-\overline{\mathbf{m}}||_2 = \sqrt{\sum_{j=0}^{s-1}\sum_{k=1}^{s}\left(m_{1(sj+k)} - \frac{n}{s^2r}\right)^2}.$$

If for each $j \in \{0, \ldots, s-1\}$,

$$\sum_{i=1}^{s} \left(m_{1(sj+i)} - \frac{n}{s^2 r} \right)^2 < \frac{R^2}{s}$$

then clearly

$$||\mathbf{x}||_2 < R.$$

Thus a necessary condition for $||\mathbf{x}||_2 \geq R$ is that

$$\sum_{i=1}^{s} \left(m_{1(sj+i)} - \frac{n}{s^2 r} \right)^2 \geq \frac{R^2}{s}$$

for at least one $j \in \{0, \ldots, s-1\}$. This results in the following upper bound on the tail of $\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2$, obtained by counting the number of ways in which h groups of elements $m_{1(sj+1)}, \ldots, m_{1(sj+s)}$ can be at distance at least $R/\sqrt{s}$ from the "central" vector having $m_i = \frac{n}{s^2 r}$ for all i:

$$C\binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \sum_{h=1}^{s} \binom{s}{h} \left(\sum_{||\mathbf{x}|| \geq \frac{R}{\sqrt{s}}} H'_n(\mathbf{m}) \right)^h \left(\sum_{||\mathbf{x}|| < \frac{R}{sqrts}} H'_n(\mathbf{m}) \right)^{s-h}.$$

■

Proof of Theorem 5.16. By Lemma 5.21 we can bound the sum above by

$$C\binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \sum_{h=1}^{s} \binom{s}{h} \left(\sum_{\mathbf{x}:||\mathbf{x}||_2 \geq \frac{\rho(n)}{\sqrt{s}}} H'_n(\mathbf{m}) \right)^h \left(\sum_{\mathbf{x}:||\mathbf{x}||_2 < \frac{\rho(n)}{\sqrt{s}}} H'_n(\mathbf{m}) \right)^{s-h}, \tag{5.72}$$

and by Lemmas 5.18 and 5.20 and noticing that the inner sums both have at most $\binom{\frac{n}{sr}}{s-1}$ terms we can bound (5.72) above by

$$C\binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}}\sum_{h=1}^{s}\binom{s}{h}\left(\binom{\frac{n}{sr}}{s-1}H_n(\mathbf{m}')\right)^h\left(\binom{\frac{n}{sr}}{s-1}H_n(\overline{\mathbf{m}})\right)^{s-h}, \quad (5.73)$$

for some $\mathbf{m}'$ with

$$||\mathbf{x}'||_2 = \rho(n)/\sqrt{s},$$

and $\overline{\mathbf{m}}$ with

$$\overline{m}_i = \frac{n}{s^2r}$$

for all i. By Lemma 5.20, $H_n(\overline{\mathbf{m}}) > H_n(\mathbf{m}')$ and so we have

$$(H_n(\overline{\mathbf{m}}))^{s-h}(H_n(\mathbf{m}'))^h = (H_n(\overline{\mathbf{m}}))^{s-1}H_n(\mathbf{m}')\times\left(\frac{H_n(\mathbf{m}')}{H_n(\overline{\mathbf{m}})}\right)^{h-1} \leq (H_n(\overline{\mathbf{m}}))^{s-1}H_n(\mathbf{m}').$$

Thus we can bound (5.73) above by

$$C\binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}}2^s\binom{\frac{n}{sr}}{s-1}^s(H_n(\overline{\mathbf{m}}))^{s-1}(H_n(\mathbf{m}')). \quad (5.74)$$

We can use Lemma 5.17 to give upper bounds on $H_n(\mathbf{m}')$ and $H_n(\overline{\mathbf{m}})$ and so bound equation (5.74) above by

$$\begin{aligned}
&C\binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}}2^s\binom{\frac{n}{sr}}{s-1}^s\left(\left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}}\left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}}\right)^{\frac{n}{s}}\frac{s^{\frac{5s}{2}}}{(s-1)^{2s}}e^{\frac{r}{(s-1)^2}}e^{\frac{sr}{12n}}\right)^{s-1}\\
&\qquad\times\left(\left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}}\left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}}\right)^{\frac{n}{s}}\frac{s^{\frac{5s}{2}}}{(s-1)^{2s}}e^{\left(\frac{s^2r^2}{(s-1)^2n}-\frac{s^2r}{2n}\right)||\mathbf{x}'||^2+\frac{r}{(s-1)^2}}e^{\frac{sr}{12n}}\right)\\
&= C\binom{n/r}{\frac{n}{sr},\ldots,\frac{n}{sr}}2^s\left(\binom{\frac{n}{sr}}{s-1}\left(\frac{sr}{2\pi n}\right)^{\frac{s-1}{2}}\left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}}\right)^{\frac{n}{s}}\frac{s^{\frac{5s}{2}}}{(s-1)^{2s}}\right)^s\\
&\qquad\times e^{\left(\frac{s^2r^2}{(s-1)^2n}-\frac{s^2r}{2n}\right)||\mathbf{x}'||^2+\frac{sr}{(s-1)^2}-\frac{sn-s^2r}{(s-1)n+sr}+\frac{s^2r}{12n}}
\end{aligned}$$

$$= C_2 \binom{n/r}{\frac{n}{sr}, \ldots, \frac{n}{sr}} \left(n^{\frac{s-1}{2}} \left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}} \right)^{\frac{n}{s}} \right)^s e^{\left(\frac{s^2 r^2}{(s-1)^2 n} - \frac{s^2 r}{2n} \right) ||\mathbf{x}'||^2}, \tag{5.75}$$

for some positive constant C_2. Finally, we use Stirling's approximations to give an upper bound on the multinomial coefficient and thus bound (5.75) above by

$$\left(\frac{r}{2\pi n} \right)^{\frac{s-1}{2}} s^{\frac{n}{r}+\frac{s}{2}} e^{\frac{r}{12n}} C_2 n^{\frac{s^2-s}{2}} \left(\frac{(s-1)^2}{s^{2-\frac{1}{r}}} \right)^n e^{\left(\frac{s^2 r^2}{(s-1)^2 n} - \frac{s^2 r}{2n} \right) ||\mathbf{x}'||^2} =$$

$$= C_2 n^{\frac{(s-1)^2}{2}} (c_{r,s})^{2n} e^{\left(\frac{s^2 r^2}{(s-1)^2 n} - \frac{s^2 r}{2n} \right) ||\mathbf{x}'||^2} \tag{5.76}$$

Since $\rho(n)$ is asymptotically larger than $\sqrt{n}$, equation (5.76) is $o(a_n^2)$ if and only if

$$\begin{aligned} \frac{s^2 r^2}{(s-1)^2 n} - \frac{s^2 r}{2n} &< 0 \\ \frac{r}{(s-1)^2} &< \frac{1}{2} \\ 2r &< (s-1)^2, \end{aligned}$$

which is true for any positive s, r with $r < \frac{s}{2} \log(s-1)$. ■

5.3.3 Upper Bounds on $\chi(R_r(\mathcal{T}_n))$

We can now complete the proof of Theorem 5.2. The Cauchy-Schwarz inequality gives us that the probability of $R_r(\mathcal{T}_n)$ having at least one proper balanced colouring can be bounded below by the ratio

$$\frac{\left(\mathbf{E} W'_{r,s}(\mathcal{T}_n) \right)^2}{\mathbf{E} W'_{r,s}(\mathcal{T}_n)^2}.$$

From Theorem 5.11 and Lemma 5.10 this is asymptotically equal to:

$$\frac{\left(s^{\frac{s}{2}}\left(\frac{se^{\frac{r-1}{s-1}}}{s-1}\right)^{s}\left(\frac{r}{2\pi}\right)^{\frac{s-1}{2}}a_n\right)^{2}}{\frac{s^{s^2+s+1}e^{\frac{s^2(r-1)}{(s-1)^2}}(s-2)^{(s-1)^2}}{(s-1)^{2s^2-2s+2}\left(1-\frac{2r}{(s-1)^2}+\frac{r}{(s-1)^4}\right)^{\frac{(s-1)^2}{2}}}\left(\frac{r}{2\pi}\right)^{s-1}(a_n)^2}$$

which simplifies to

$$\frac{e^{\frac{s(s-2)(r-1)}{(s-1)^2}}\left(r-2r(s-1)^2+(s-1)^4\right)^{\frac{(s-1)^2}{2}}}{s^{(s-1)^2}(s-2)^{(s-1)^2}}.$$

■

Table 5.4 gives numerical values for this lower bound for early values of s and r.

$r \setminus s$	5	6	7	8	9	10	11	12	13	14	15	16	17
3	0.495	—	—	—	—	—	—	—	—	—	—	—	—
4	—	0.448	0.594	—	—	—	—	—	—	—	—	—	—
5	—	—	0.424	0.549	0.641	—	—	—	—	—	—	—	—
6	—	—	0.273	0.419	0.518	0.603	0.669	—	—	—	—	—	—
7	—	—	—	0.285	0.400	0.496	0.574	0.636	0.687	—	—	—	—
8	—	—	—	—	0.294	0.394	0.479	0.551	0.610	0.659	0.700	—	—
9	—	—	—	—	0.205	0.301	0.390	0.466	0.532	0.588	0.635	0.676	0.710

Table 5.4: Numerical values for $\frac{\left(\mathbf{E}W'_{r,s}(\mathcal{T}_n)\right)^2}{\mathbf{E}W'_{r,s}(\mathcal{T}_n)^2}$ giving a lower bound on the probability that there exists at least one proper balanced s-colouring for $5 \leq s \leq 17$, $3 \leq r \leq 9$.

5.4 Comparisons with Other Models of Random Graphs

In this chapter we have given bounds on the proportion of trees reducing to graphs that can be properly s-coloured by calculating the first two moments of the number of proper s-colourings of a random tree with empires formed

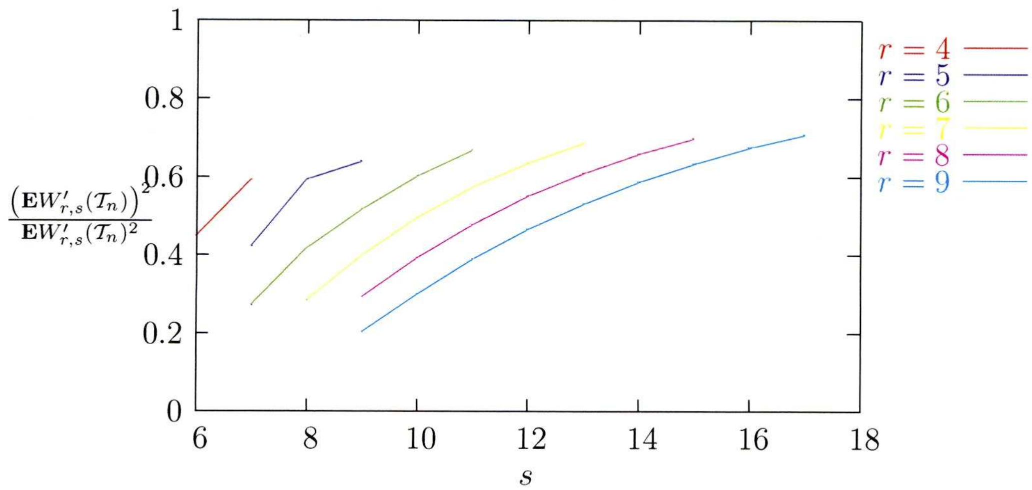

Figure 5.5: Lower bounds on the probability that there exists at least one proper balanced s-colouring for $6 \leq s \leq 17$, $4 \leq r \leq 9$

by exactly r empires each. We conclude this chapter by quickly comparing our treatment with similar results in the literature.

Because of the similarities highlighted in Section 3.5 between $R_r(\mathcal{T}_n)$ and $G(\frac{n}{r}, n-1)$, and $\mathcal{G}_{\frac{n}{r},2r}$ it is natural to compare the results on $\chi(R_r(\mathcal{T}_n))$ in this thesis with those proved for other types of random graphs. The state of the art there is represented by the work of Achlioptas and Moore (on random regular graphs of small degree) [4] and, later, Achlioptas and Naor [5] for Erdős-Rényi random graphs of small average degree. In this chapter it was shown that $R_r(\mathcal{T}_n)$ is a.a.s. not s_r-colourable, where s_r is the largest integer such that

$$\left(\frac{s_r - 1}{s_r^{1-\frac{1}{r}}}\right) < 1,$$

but for any (fixed) s, r such that

$$r < \frac{s}{2}\log(s-1),$$

there is at least a positive constant probability of s-colourability. Roughly speaking this result implies that, for sufficiently large (but fixed) values of r, with high probability, $\chi(R_r(\mathcal{T}_n))$ is larger than about $r/\log r$ but, with positive probability, smaller than about twice such quantity. A result obtained by Frieze and Luczak [32] shows that, for r large enough, the chromatic number of a random regular graph of degree $2r$ is a.a.s. $\frac{r}{\log 2r}$ (and similar results [52] exist about $\chi(G(n,p))$. This implies that the likely range of values for $\chi(\mathcal{G}_{\frac{n}{r},2r})$ lays is an interval of size $o(r/\log r)$. Furthermore, using rather non-trivial analytical techniques, Achlioptas *et al.* [4, 5] proved that for both Erdős-Rényi random graphs with constant average degree, and for random regular graphs of constant degree, this *concentration interval* is actually restricted to a very small range of up to three different values.

Thus, although our results support the claim that $2r$ is indeed a rather weak estimate for the typical value of $\chi(R_r(\mathcal{T}_n))$, they are much weaker than analogue statements obtained for other related random graph models. Given that the techniques used in this thesis are relatively simple, the question of whether our results can be tightened up to qualitatively mirror the results mentioned above, is the major open problem of our work.

Chapter 6

Conclusions

In this thesis we examined the empire-colourability of trees. For arbitrary trees we showed that if the vertex set is partitioned into empires of size exactly $r \geq 1$, then $2r$ colours are always sufficient to give a proper colouring, and furthermore there exist trees that require this many colours.

In Chapter 3 we gave a precise definition of the problem being studied and defined the concept of the r-reduced graph of a given graph. We then studied a number of properties of the r-reduced graphs of trees such as vertex degrees, connectivity, and the presence of certain small subgraphs.

In Chapter 4 we studied three algorithms for graph colouring. It was shown that for all positive r there exists an algorithm that can properly colour any tree with empires consisting of r vertices, using at most $2r$ colours. The actual results of the algorithms were in fact much better than this, suggesting that this upper bound is quite pessimistic.

In Chapter 5 we gave a precise characterisation of the first two moments of a random variable counting the number of s-empire colourings of a random tree with vertex set partitioned into empires of size r. From this we were able to give upper and lower bounds on the probability that a random tree has at least one r-empire s-colouring. We found that for any $r \geq 2$, there exists a number

$$s_r = \left\lceil \frac{r}{\log r} \right\rceil \left(1 + O\left(\frac{1}{\log \log r}\right)\right),$$

such that any random tree on empires of size r a.a.s. admits no proper s-colouring for any $s \leq s_r$. Finally, for any s and r such that

$$r < \frac{s}{2} \log(s - 1),$$

we were able to show that there is at least a constant positive probability that a random tree has at least one r-empire s-colouring.

A number of questions are left open by this report. From the average-case analysis point of view the main open issue is the colourability of random trees given fixed values of s and r such that $s > s_r$ and $r > \frac{s}{2} \log(s - 1)$. In this range we can neither confirm that a random tree a.a.s. has no proper s-colouring, nor give any positive lower bound on the probability of such a colouring. Furthermore, while we have found exact expressions for all moments of $W_{r,s}(\mathcal{T}_n)$ and $W'_{r,s}(\mathcal{T}_n)$, we have only found asymptotically tight expressions for the first moment of each and the second moment of $W'_{r,s}(\mathcal{T}_n)$. A full asymptotic characterization of all moments of these random variables should be possible and could be the subject of further investigation. Finally, could the methods used in this thesis be applied to the r-empire colourability of $G(n, p)$, when p is such that $G(n, p)$ is a.a.s. planar? For sufficiently small p the graph is likely to be quite simple, consisting of a number of trees and unicyclic graphs and so many of the methods used in the analysis of trees may also apply here.

Even more problems remain unsolved from the worst-case analysis point of view. First of all we may ask if it is NP-hard to find the 2-empire chromatic number of a tree? We know that any tree a.a.s. has 2-empire chromatic number of either 3 or 4, but so far there are no bounds on the probability of a given tree admitting a proper 2-empire 3-colouring. Next, could we find some structural characterisation of the class of r-reduced graphs of trees? The results in Chapter 3 suggest that all such graphs share a number of properties such as having average degree $2r - \frac{2r}{n}$, minimum degree r and being connected, but it is not enough to say that any graph with these properties is always the r-reduced graph of some tree. Finding such a characterisation would mean that the problem of r-empire colouring of trees would be reduced to that of graph colouring on a certain family of graphs.

Of course, it may be interesting to study other variations of the empire colouring problem. For instance, one could set a lower bound on the distance in $\mathcal{T}_n$ between two vertices from the same empire. This would remove the issue of loops in the reduced graph and may also reduce the number of short cycles. It is possible that such a restriction would reduce the number of colours required.

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Index